Topology Supplement

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Contents

These notes serve as an introduction to some topological concepts and a supplement to what you have learned in class. If you'd like to learn topology more thoroughly, I highly recommend the book *General Topology* by Ryszard Engelking. There is a discussion of many, many interesting topological concepts there, whereas we just scratch the surface.

1 Topological Spaces

Definition 1.1. A *topological space* is a pair (X, τ) consisting of a set X and $\tau \subseteq \mathcal{P}(X)$ such that:

(i) $\varnothing \in \tau$ and $X \in \tau$.

(ii) For any $U, V \in \tau$, we have $U \cap V \in \tau$.

(iii) For any set index set A and $\{U_\alpha\}_{\alpha \in A} \subseteq \tau$, we know that $\bigcup\limits_{\alpha \in A} U_\alpha \in \tau$.

We usually call τ the *topology* on X and the elements of τ the *open sets* in X.

Example 1.2. For any set X, $\{\varnothing, X\}$ is a topology on X.

Example 1.3. For any set X, $\mathcal{P}(X)$ is a topology on X.

Example 1.4. For any set X, the set

$$\{E \subseteq X \mid E \text{ is finite or } E^{\mathsf{c}} \text{ is finite}\}$$

is a topology on X.

Example 1.5. The open subsets of $\mathbb{R}^n$, i.e. the set

$$\tau = \{U \subseteq \mathbb{R}^n \mid \text{ for all } x \in U \text{ there exists } \varepsilon > 0 \text{ such that } B(x, \varepsilon) \subseteq U\}$$

is a topology on $\mathbb{R}^n$.

Example 1.5 is probably the most important for you because it is the one we work with in this class. However, the study of topology is remarkably generic and covers a wide array of structures, so the results we do here apply in many places. Additionally, although we are working with $\mathbb{R}^n$, the isolation of the study of the topological structure can be helpful as it points to the key properties you use and the generality can give you a new appreciation of how $\mathbb{R}^n$ works.

Definitions 1.6. We fix a topological space (X, τ).

(a) $C \subseteq X$ is *closed* if $C^{\mathsf{c}} = X \setminus C \in \tau$ i.e. if C^{c} is open.

(b) For $A \subseteq X$, we say that $N \subseteq X$ is a *neighborhood of* A if there exists an open set $U \in \tau$ with $A \subseteq U \subseteq N$. We denote the set of all neighborhoods of A as $\mathcal{N}(A)$. When $A = \{x\}$, we usually write $\mathcal{N}(x)$ for $\mathcal{N}(\{x\})$.

(c) Let $E \subseteq X$. Then for $x \in X$,

- if E is a neighborhood for x, we say that x is an *interior point* of E. The set of all interior points of E is denoted as $\operatorname{int}(E)$ or E°.
- if E^{c} is a neighborhood for x, we say that x is an *exterior point* of E. The set of all exterior points is denoted as $\operatorname{ext}(E)$.
- if neither E nor E^{c} is a neighborhood for x, we say that x is a *boundary point* of E. The set of boundary points of E is denoted as ∂E.
- if for every $U \in \mathcal{N}(X)$ we know that $(U \setminus \{x\}) \cap E$ is nonempty, we say that x is a *limit point* of E. The set of limit points of E is denoted as E'.

In addition, we define $\overline{E} = E^\circ \cup \partial E$.

Remark 1.7. Closed sets is not a great term as slightly ridiculously nothing prevents a set from being both open and closed. Perhaps a better term for closed sets would be "co-open" but I suppose that would be equally ridiculous.

Exercise 1.8. Show for a topological space (X, τ) that $\varnothing$ and X are closed, that the finite unions of closed sets are closed, and that arbitrary intersections of closed sets are closed.

Exercise 1.9. Show that $\overline{E} = \{x \in X \mid \text{ for all } U \in \mathcal{N}(x), U \cap E \neq \varnothing\}$.

Exercise 1.10. For $E \subseteq X$, show that every $x \in X$ is exactly one of an interior point, exterior point, or boundary point.

Exercise 1.11. Show that these definitions coincide with the ones you were given in $\mathbb{R}^n$ with its usual topology.

Proposition 1.12. *Let (X, τ) be a topological space and $E \subseteq X$. Then the following hold:*

(a) E° is open, $\overline{E}$ is closed, and ∂E is closed.

(b) E is open if and only if $E = E^\circ$.

(c) $\overline{E} = E \cup E'$.

Remark 1.13. Note we have excluded that E' is closed here. This is because this can be false in general. Consider $X = \{0, 1\}$ with $\tau = \{\varnothing, X\}$. Then $\{0\}' = \{1\}$, and $\{1\}$ is not closed.

Proof. To show E° is open, suppose it is nonempty and let $x \in E^\circ$. Then we note there is some $U_x \in \tau$ such that $x \in U_x \subseteq E$. Furthermore, we in fact have $U_x \subseteq E^\circ$ - for every $y \in U_x$, we know $y \in U_x \subseteq E$ and so E is a neighborhood for y, i.e. $y \in E^\circ$. Thus

$$E^\circ = \bigcup_{x \in E^\circ} U_x$$

and so E° is a union of open sets and thus open.
Now we note $\overline{E} = \left((E^{\mathsf{c}})^\circ\right)^{\mathsf{c}}$ (exercise) and so $\overline{E}$ is the complement of an open set and thus open. Then $\partial E = \overline{E} \cap \overline{E^{\mathsf{c}}}$ as an intersection of closed sets is closed.
For (b), if E is open then E is clearly a neighborhood for every $x \in E$ and so $E = E^\circ$. On the other hand, if $E = E^\circ$, then we know E° is open and thus E must be as well.
For (c), it is clear that $E \subseteq \overline{E}$ and that $E' \subseteq \overline{E}$ by Exercise 1.9. On the other hand, if $x \in \overline{E} \smallsetminus E$, then again by Exercise 1.9 we know that for every neighborhood U of x, $\varnothing \neq U \cap E = (U \smallsetminus \{x\}) \cap E$ and so $x \in E'$. ∎

2 Bases and Subbases

Rather than working with the whole topology, it is often useful to work with a subset that somehow captures all the necessary information. This motivates the next definitions.

Definitions 2.1. Let (X, τ) be a topological space and $x \in X$. We say $\mathcal{B}_x \subseteq \tau$ is a *neighborhood base* for x (with respect to τ) if

- for each $V \in \mathcal{B}_x$, we have that $x \in V$, and
- for each $U \in \tau$ with $x \in U$, there is some $V \in \mathcal{B}_x$ such that $V \subseteq U$.

A *base* $\mathcal{B}$ for τ is some $\mathcal{B} \subseteq \tau$ such that $\mathcal{B}$ contains a neighborhood base for each $x \in X$. Equivalently, $\mathcal{B} \cap \mathcal{N}(x)$ is a neighborhood base for x for each $x \in X$.

Proposition 2.2. *Let (X, τ) be a topological space and suppose $\mathcal{B} \subseteq \tau$. Show that $\mathcal{B}$ is a base for τ if and only if for all $U \in \tau$ there exists some $\mathcal{B}_U \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{B}_U$.*

Proof. Firstly suppose $\mathcal{B}$ is a base for τ and let $U \in \tau$. Set $\mathcal{B}_U = \{B \in \mathcal{B} \mid B \subseteq U\}$. We then note that clearly $U \supseteq \bigcup \mathcal{B}_U$. For the reverse inclusion, let $x \in U$. Since $\mathcal{B}$ is a base, we can find some $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$, so $B \in \mathcal{B}_U$ which tells us $x \in \bigcup \mathcal{B}_U$.
For the opposite direction, let U be open and let $x \in U$. As in the statement, pick $\mathcal{B}_U \subseteq \mathcal{B}$ with $U = \bigcup \mathcal{B}_U$. Then $x \in B$ for some $B \in \mathcal{B}_U$, and we clearly have $B \subseteq U$, so we have that $\mathcal{B}$ is a base as desired. ■

Proposition 2.3. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$. Then $\mathcal{B}$ is a base for some (necessarily unique) topology τ on X if and only if $X = \bigcup \mathcal{B}$ and for every $U, V \in \mathcal{B}$ and $x \in U \cap V$, there exists some $W \in \mathcal{B}$ such that $x \in W$ and $W \subseteq U \cap V$.*

Proof. If $\mathcal{B}$ is a base, then clearly $X = \bigcup \mathcal{B}$. Further, for any $U, V \in \mathcal{B} \subseteq \tau$, we know $U \cap V \in \tau$. Thus we can find some $W \in \mathcal{B}$ with $x \in W$ and $W \subseteq U \cap V$ as desired.
For the opposite direction, define

$$\tau = \{U \subseteq X \mid \text{ for all } x \in U \text{ there exists } V \in \mathcal{B} \text{ with } x \in V \text{ and } V \subseteq U\}.$$

It's obvious that $\varnothing \in \tau$ and we see that $X \in \tau$ because $X = \bigcup \mathcal{B}$. It is also clear that τ is closed under unions. To show that τ is closed under finite intersections, let $U, V \in \tau$. If $U \cap V = \varnothing$, then we know $U \cap V \in \tau$, so suppose $U \cap V$ is nonempty and let $x \in U \cap V$. Then we can find $\tilde{U}, \tilde{V} \in \mathcal{B}$ with $\tilde{U} \subseteq U$, $\tilde{V} \subseteq V$, and $x \in \tilde{U} \cap \tilde{V}$. Then by our condition, we can find $W \in \mathcal{B}$ such that $x \in W$ and $W \subseteq \tilde{U} \cap \tilde{V} \subseteq U \cap V$. Our choice of $x \in U \cap V$ was arbitrary, so we conclude that $U \cap V \in \tau$ as desired. ■

Exercise 2.4. Prove that if $\mathcal{B}$ is a base for topologies τ_0 and τ_1 on X, then $\tau_0 = \tau_1$. This justifies the "necessarily unique" portion of the propostion.

Corollary 2.5. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$. Suppose that $X \in \mathcal{B}$ and $\mathcal{B}$ is closed under finite intersections, that is for any $A, B \in \mathcal{B}$ we know that $A \cap B \in \mathcal{B}$. Then $\mathcal{B}$ is a base for some topology τ on X.*

Exercise 2.6. Let X be a set and $\{\tau_i\}_{i \in I}$ be collection of topologies on X. Show that $\bigcap\limits_{i \in I} \tau_i$ is also a topology on X.

Definitions 2.7. Let X be a set and let $\mathcal{E} \subseteq \mathcal{P}(X)$. We define the *topology generated by $\mathcal{E}$* to be the set

$$\tau(\mathcal{E}) = \bigcap \{\tau \subseteq \mathcal{P}(X) \mid \tau \text{ is a topology on } X \text{ and } \mathcal{E} \subseteq \tau\}.$$

We note this is well-defined in light of the previous exercise and the fact that $\mathcal{P}(X)$ is always a topology on X containing $\mathcal{E}$.
If (X, τ) is a topological space and $\mathcal{E} \subseteq \tau$ with $\tau = \tau(\mathcal{E})$, we say $\mathcal{E}$ is a *subbase* for τ.

Proposition 2.8. *Let X be a set and $\mathcal{E} \subseteq \mathcal{P}(X)$. Set*

$$\tilde{\tau}(\mathcal{E}) = \{X\} \cup \left\{ \bigcup_{\alpha \in A} \left(\bigcap_{k=1}^{n} E_k^{(\alpha)} \right) \;\middle|\; E_k^{(\alpha)} \in \mathcal{E} \textit{ for each } 1 \leq k \leq n \textit{ and each } \alpha \in A \right\}.$$

Then $\tau(\mathcal{E}) = \tilde{\tau}(\mathcal{E})$. In other words, the topology generated by $\mathcal{E}$ consists of all unions of finite intersections of elements of $\mathcal{E}$ along with $\varnothing$ and X.

Proof. We note any topology containing $\mathcal{E}$ must certainly contain $\varnothing$, X, and all unions of finite intersections of elements of $\mathcal{E}$, so it suffices to check that $\tilde{\tau}(\mathcal{E})$ is a topology. We see that the set

$$\mathcal{E}' = \{X\} \cup \left\{ \bigcap_{k=1}^{n} E_k \;\middle|\; E_k \in \mathcal{E} \text{ for each } 1 \le k \le n \right\}$$

is clearly closed under finite intersections and contains X, so by Corollary 2.5 we know that $\mathcal{E}'$ is a base for some topology τ. Proposition 2.2 implies that τ is precisely $\tilde{\tau}(\mathcal{E})$, giving the desired result. ■

Exercise 2.9. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $\mathcal{E} \subseteq \tau_Y$ be a subbase for Y. Show that $f : X \to Y$ is continuous iff $f^{-1}[E] \in \tau_X$ for all $E \in \mathcal{E}$.

3 Continuous Maps

Definition 3.1. Given topological space (X, τ_X) and (Y, τ_Y) along with a function $f : X \to Y$, we say that f is continuous if $f^{-1}[U] \in \tau_X$ for all $U \in \tau_Y$.

Exercise 3.2. Given (X, τ_X), (Y, τ_Y), and (Z, τ_Z) topological space and continuous maps $f : X \to Y$ and $g : Y \to Z$, the map $g \circ f : X \to Z$ is also continuous.

Once you finish Exercise 3.2, note that the proof is shorter than in the $\mathbb{R}^n$ case (even though they have essentially the same content).

Exercise 3.3. Show that $f : X \to Y$ is continuous if and only if $f^{-1}[C]$ is closed in X for every closed C in Y.

Exercise 3.4. Show for $E \subseteq \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ that f is (ε-δ) continuous if and only if for every $z \in E$ and every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$B(z, \delta) \cap E \subseteq f^{-1}[B(f(z), \varepsilon)].$$

Use this to show that f is (ε-δ) continuous if and only if f is continuous as a map from E with the subspace topology to $\mathbb{R}^m$.

Exercise 3.5. Let X and Y be topological spaces. Show that $f : X \to Y$ is continuous if and only if $f\left[\overline{A}\right] \subseteq \overline{f[A]}$.

Remark 3.6. A topology can also be defined purely in terms of a closure operator, and Exercise 3.5 is the way to phrase continuity in this language.

4 Constructing Topological Spaces

Given some topological spaces, it will often be useful for us to investigate ways to make new useful topological spaces. We explore some constructions here.

4.1 Subspace Topologies

Definition 4.1. Given a topological space (X, τ) and $E \subseteq X$, we define a topology τ_E on E by setting

$$\tau_E = \{V \subseteq X \mid V = U \cap E \text{ for some } U \in \tau\}$$

i.e. the set of all intersections of E with open sets of X. We call τ_E the *subspace topology* on E. Sometimes when talking about both τ and τ_E at the same time, we say $U \in \tau_E$ if U is *relatively open* in E. You naturally get a similar definition for relatively closed.

In the subspace topology on $\mathbb{Q}$ inherited from our standard topology on $\mathbb{R}$, we see that no nonempty relatively open subset of $\mathbb{Q}$ is open in $\mathbb{R}$, and no open subset of $\mathbb{R}$ is relatively open in $\mathbb{Q}$. These are two related but different topologies.

Exercise 4.2. If (X, τ_X) and (Y, τ_Y) are topological spaces with $f : X \to Y$ continuous and $E \subseteq X$ given the subspace topology. Show that the restriction $f\restriction_E : E \to Y$ is also continuous.

4.2 Product Topologies

Definition 4.3. Let I be some set and $\{(X_i, \tau_i)\}_{i \in I}$ be a collection of topological spaces. We define

$$\mathcal{B}_I = \left\{ \prod_{i \in I} U_i \subseteq \prod_{i \in I} X_i \;\middle|\; U_i \in \tau_i \text{ for all } i \in I \text{ and } U_i = X_i \text{ for cofinitely many } i \in I \right\}$$

Exercise 4.4. Show that $\mathcal{B}_I$ as defined above is a base for some topology τ on $\prod_{i \in I} X_i$.

In light of the previous exercise, we define the *product topology* on $\prod_{i \in I} X_i$ to be the topology generated by $\mathcal{B}_I$. Note that the product topology is generated by the subbbase

$$\left\{ \prod_{i \in I} U_i \;\middle|\; U_i \in \tau_i \text{ for all } i \in I \text{ and there exists a unique } i \in I \text{ such that } U_i \neq X_i \right\}$$
$$= \bigcup_{i \in I} \left\{ \pi_i^{-1}[U] \;\middle|\; U \in \tau_i \right\}$$

where π_i is the projection map onto the i-th coordinate. Whenever we have a product of topological spaces, we will assume they are endowed with the product topology unless stated otherwise.

Exercise 4.5. Show that the product topology is the smallest topology on $\prod_{i \in I} X_i$ such that the projection map $\pi_i : \prod_{i \in I} X_i \to X_i$ is continuous for all $i \in I$. In other words, prove

- π_i is continuous for all $i \in I$ with respect to the product topology, and
- if τ is any topology on $\prod_{i \in I} X_i$ such that π_i is continuous for all $i \in I$, then τ contains the product topology

Proposition 4.6. *Let I be some set and $\{(X_i, \tau_i)\}_{i\in I}$ be a collection of topological spaces and endow $\prod_{i\in I} X_i$ with the product topology. Let Z be a topological space and $g : Z \to \prod_{i\in I} X_i$ be any function. Then g is continuous iff $\pi_i \circ g$ is continuous for each $i \in I$.*

Proof. If g is continuous, then clearly $\pi_i \circ g$ is continuous for all $i \in I$ as a composition of continuous maps. On the other hand, suppose $\pi_i \circ g$ is continuous for each $i \in I$. In light of Exercise 2.9, it suffices to show that $g^{-1}[\pi_i^{-1}[U]]$ is open for each $i \in I$ and $U \in \tau_i$. But we note that $g^{-1}[\pi_i^{-1}[U]] = (\pi_i \circ g)^{-1}[U]$, which is open by our hypothesis that $\pi_i \circ g$ is continuous as desired. ■

Exercise 4.7. Let X be a topological space and I be a set. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in X^I. Show that $(f_n)_{n\in\mathbb{N}}$ converges in X^I with the product topology iff $(f_n(i))_{n\in\mathbb{N}}$ is a convergent sequence in X for each $i \in I$.

5 Nets

In general topological spaces, sequences often aren't enough to characterize the topology. We present some examples of this here.

Exercise 5.1. Consider $\mathbb{R}$ with the cocountable topology τ, that is

$$\tau = \{A \subseteq \mathbb{R} \mid A^{\mathsf{c}} \text{ is countable}\} \cup \{\varnothing\}.$$

Prove the following:

(a) Every convergent sequence in $(\mathbb{R}, \tau)$ is eventually constant.

(b) Show that $[0, 1] \subseteq \mathbb{R}$ is dense but for every convergent sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in [0, 1]$ for all $k \in \mathbb{N}$, we have that $\lim_k x_k \in [0, 1]$.

(c) For every topological space Y, any function $f : (\mathbb{R}, \tau) \to Y$ is sequentially continuous.

(d) There is some topological space Y and function $f : (\mathbb{R}, \tau) \to Y$ such that f is not continuous.

Exercise 5.2. Consider the topological space $\mathbb{R}^{\mathbb{R}}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ endowed with the product topology and the subspace $C(\mathbb{R}; \mathbb{R}) \subseteq \mathbb{R}^{\mathbb{R}}$. Show that $C(\mathbb{R}; \mathbb{R})$ is dense in $\mathbb{R}^{\mathbb{R}}$ but there exists a function $f \in \mathbb{R}^{\mathbb{R}}$ such that there is no sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$. (a bit tricky)

This motivates the definition of something more generic than a sequence. For this, we need to index with something other than $\mathbb{N}$. We define this object now.

Definition 5.3. Let A be a set and $\lesssim$ be a relation on A. We say that $(A, \lesssim)$ is a *directed set* if

- $\alpha \lesssim \beta$ and $\beta \lesssim \alpha \Leftrightarrow \alpha = \beta$ for each $\alpha, \beta \in A$,
- $\alpha \lesssim \beta$ and $\beta \lesssim \gamma$ implies $\alpha \lesssim \gamma$ for each $\alpha, \beta, \gamma \in A$, and
- for each $\alpha, \beta \in A$, there exists some $\gamma \in A$ such that $\alpha \lesssim \gamma$ and $\beta \lesssim \gamma$.

Examples 5.4. We list some important examples of directed sets now.

- $(\mathbb{N}, \leq)$ is a directed set.
- More generally, for any ordinal κ, we know that $(\kappa, \leq)$ is a direct set.
- For any topological space X and $x \in X$, the set $(\mathcal{N}(x), \supseteq)$ is a directed set.
- For any two directed sets $(A, \lesssim_A)$ and $(B, \lesssim_B)$, define the relation $\lesssim$ on $A \times B$ by $(\alpha, \beta) \lesssim (\alpha', \beta')$ iff $\alpha \lesssim_A \alpha'$ and $\beta \lesssim_B \beta'$. Then $(A \times B, \lesssim)$ is a directed set.

Definition 5.5. Let (X, τ) be a topological space. A *net* is any function $(x_\alpha)_{\alpha \in A}$ from some directed set A into X. We say that x_α converges to x for some $x \in X$ if for every $U \in \mathcal{N}(x)$ there exists some $\beta \in A$ such that for all $\alpha \geq \beta$ we have that $x_\alpha \in U$. In this case, we say $\lim\limits_{\alpha \in A} x_\alpha = x$.

Exercise 5.6. Let (X, τ) and be a topological space. Prove the following:

(a) For any $E \subseteq X$, we have that $x \in \overline{E}$ if and only if there exists some net $(x_\alpha)_\alpha$ with $x_\alpha \in E$ for all $\alpha \in A$ such that $\lim\limits_{\alpha \in A} x_\alpha = x$.

(b) For any topological space Y and function $f : X \to Y$, we have that f is continuous if and only if for every convergent net $(x_\alpha)_\alpha$ in X converging to some x, we have that $\lim\limits_{\alpha \in A} f(x_\alpha) = f(x)$.

Exercise 5.6a shows that nets really do characterize the topology by a closure operation.

6 Compactness

Definition 6.1. Let (X, τ) be a topological space. We say that $K \subseteq X$ is (topologically) *compact* if for every set of open sets $\mathcal{U} \subseteq \tau$ such that

$$\bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} U \supseteq K$$

there exists $U_1, \ldots, U_n \in \mathcal{U}$ such that $\bigcup\limits_{k=1}^{n} U_k \supseteq K$. This is sometimes stated succinctly as "every open cover has a finite subcover".

It really might seem like I'm just bombarding you with topology definitions, but I promise you this one is important.

It is hard to overstate how useful compactness is, even in such an abstract form. In some sense, it allows us to turn difficult questions of infinite control into a question of finite control.

Proposition 6.2. *Every topologically compact subset of $\mathbb{R}^n$ is bounded.*

Proof. Let $K \subseteq \mathbb{R}^n$ be topologically compact. Then note that $\{B(x,1)\}_{x\in K}$ is a set of open sets and
$$K \subseteq \bigcup_{x\in K} B(x,1).$$
Then there exists $x_1,\ldots,x_k \in K$ such that
$$K \subseteq \bigcup_{k=1}^{n} B(x_k,1).$$
The finite unions of bounded sets are bounded and subsets of bounded sets are bounded (exercise) so K is bounded as desired. ■

Exercise 6.3. Let (X,τ) be a topological space and suppose that $K \subseteq X$ is compact. Suppose that $C \subseteq K$ is closed in X. Show that C is also compact.

Exercise 6.4. Show that every topologically compact subset of $\mathbb{R}^n$ is closed.

Definition 6.5. Let (X,τ) be a topological space. We say that X is *sequentially compact* if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X, there exists a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$.

7 Properties of Topological Spaces

7.1 Countability Axioms

Definitions 7.1. Let (X,τ) be a topological space. We say that X is *first countable* if every point has a countable neighborhood base. We say that X is *second countable* if X has a countable base. We say that X is *separable* if X has a countable dense subset.

Exercise 7.2. Suppose that X is first/second countable. Then show for any $E \subseteq X$ that E is first/second countable respectively.

Exercise 7.3. Suppose that X is second countable. Show that X is separable.

Proposition 7.4. *Suppose that X is second countable. Then X is* Lindelöf, *that is for every open cover $\{U_i\}_{i\in I}$ of X, there exists $\{U_n\}_{n\in\mathbb{N}} \subseteq \{U_i\}_{i\in I}$ such that $X = \bigcup_{n\in\mathbb{N}} U_n$.*

Proof. Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable base for X and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. For each $n \in \mathbb{N}$ for which it's possible, pick some $U_n \in \mathcal{U}$ such that $B_n \subseteq U_n$. Then the set $\mathcal{U}'$ of all such U_n is countable. We now wish to show it covers X. Let $x \in X$ and pick some $U \in \mathcal{U}$ such that $x \in U$. Since U is open, we can find some $B_n \subseteq U$ with $x \in B_n$. Then $U_n \supseteq B_n$ also contains x, showing the desired result. ■

Exercise 7.5. Suppose that X is first countable and Y is a topological space. Show that $f : X \to Y$ is continuous if and only if f is sequentially continuous.

Exercise 7.6. Suppose that X is first countable and compact. Show that X is sequentially compact.

7.2 Separation Axioms

Definition 7.7. Suppose (X, τ) is a topological space and A, B are subsets of X. We say that A and B are *separated by neighborhoods* if there exists $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$ with $U \cap V = \varnothing$. If A or B are singletons, we usually drop the set brackets.

Definition 7.8. Suppose (X, τ) is a topological space. We say that X is *Hausdorff* iff for every $x, y \in X$ such that $x \neq y$, we have x and y are separated by neighborhoods.

It's worth noting that Hausdorff implies singletons are closed.

Definition 7.9. Suppose (X, τ) is a topological space. We say that X is *normal* iff X is Hausdorff and for closed sets $A, B \subseteq X$ such that $A \cap B = \varnothing$, we know that A and B are separated by neighborhoods.

These so-called "separation axioms" are also sometimes known as T_2 and T_4 respectively. I'm not sure which pair of names is less offensive.

Remark 7.10. Of course this might make you wonder what other T_n there are for suitable n. If you'd like to be entertained/confused/frustrated/educated, feel free to read more about some other completely perfectly normal regular separation axioms.

Exercise 7.11. Prove that every metric space is normal.

Exercise 7.12. Prove that every compact Hausdorff space is normal. Hint: First prove that if K is compact and $x \notin K$, then x and K are separated by neighborhoods.

8 Uniformities

8.1 Topological Groups

Oftentimes, we will find that the topological structure of a space interacts nicely with the algebraic structure. An example of this is $\mathbb{R}^n$ when considered as a group equipped with addition. We note that the addition map add : $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and inversion maps inv : $\mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\text{add} : (x, y) \mapsto x + y$$
$$\text{inv} : x \mapsto -x$$

are both continuous. Indeed, although we take it for granted, we often exploit this fact when doing analysis in $\mathbb{R}^n$. We abstract this concept to the notion of a topological group.

Definition 8.1. Let G be a group and τ a topology on G. We say that (G, τ) is a topological group if the functions $(g, h) \mapsto gh$ from $G \times G \to G$ and $g \mapsto g^{-1}$ from $G \to G$ are continuous.

Exercise 8.2. Let G be a topological group. Then show the following hold.

(a) If $U \subseteq G$ is open, then gU, Ug, and U^{-1} are open for each $g \in G$.

(b) For every $U \in \mathcal{N}(e)$, there is some $V \in \mathcal{N}(e)$ such that $V = V^{-1}$ and $V^2 \subseteq U$.

(c) If $H \leq G$, then $\overline{H} \leq G$.

(d) If $H \leq G$ has nonempty interior, then H is clopen.

Definition 8.3. Let G be a topological group and $f : G \to \mathbb{R}$. We define the *left and right translates* of f by

$$L_y f(x) = f(y^{-1}x) \text{ and } R_y f(x) = f(xy).$$

The reason for defining the translates in this way are so that $L_{yz} = L_y L_z$ and $R_{yz} = R_y R_z$.

We can rephrase uniform continuity for functions in $\mathbb{R}$ in terms of the translates. Namely, we have that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if and only if $\left\|L_y f - f\right\|_{\sup} \to 0$ (or $\left\|R_y f - f\right\|_{\sup} \to 0$) as $y \to 0$. In other words uniform continuity tests how the natural group action on functions by translation behaves as you approach the identity. We abstract this notion to topological groups.

Definition 8.4. Let G be a topological group. We say a function $f : G \to \mathbb{R}$ is left (respectively right) uniformly continuous if $\left\|L_y f - f\right\|_{\sup} \to 0$ (respectively $\left\|R_y f - f\right\|_{\sup} \to 0$) as $y \to e$.

Proposition 8.5. *Suppose that G is a topological group and d is a metric on G compatible with the topology. Further suppose that d is* right-invariant, *that is $d(x, y) = d(xz, yz)$ for all $x, y, z \in G$. Then $f : G \to \mathbb{R}$ is left uniformly continuous (as in Definition 8.4) if and only if $f : (G, d) \to \mathbb{R}$ is (metric) uniformly continuous.*

Proof. Since d is right-invariant, we note that $d(e, y) < \delta \Leftrightarrow d(y^{-1}, e) < \delta \Leftrightarrow d(y^{-1}x, x) < \delta$, so in particular $y \in B(e, \delta)$ if and only if $y^{-1}x \in B(x, \delta)$. With this in mind, we see

$$\begin{aligned}
&f : G \to \mathbb{R} \text{ uniformly continuous} \\
\Leftrightarrow &\sup_{x \in G}\left|f(y^{-1}x) - f(x)\right| \to 0 \text{ as } y \to e \\
\Leftrightarrow &\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in X\ \forall y \in B(e, \delta), \left|f(y^{-1}x) - f(x)\right| < \varepsilon \\
\Leftrightarrow &\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in X\ \forall z \in B(x, \delta), |f(z) - f(x)| < \varepsilon \\
\Leftrightarrow &f : G \to \mathbb{R} \text{ metric uniformly continuous}
\end{aligned}$$

■

This ability to "translate" open neighborhoods of the identity to open neighborhoods anywhere is precisely what gives us the ability to define a "uniform" way to measure closeness. In the same way, a metric gives us a uniform way to measure closeness. However, it turns out, we don't even need a group structure or a metric structure to define this uniform way to test the topology. The most generic category in which uniform continuity makes sense is that of uniform spaces.

Exercise 8.7. Prove that $+$ is associative. It is not in general commutative, as so many of you angrily pointed out.

Definitions 8.6. Let X be a set. We define the following operations for $A, B \subseteq X \times X$:

$$A + B = \{(x,z) \in X \times X \mid \exists y \in X \text{ such that } (x,y) \in A \text{ and } (y,z) \in B\}$$

$$-A = \{(y,x) \in X \times X \mid (x,y) \in A\}.$$

From here, define $1A = A$ and inductively $nA = (n-1)A + A$. Also, set $\Delta_x = \{(x,x) \mid x \in X\}$. Let $\mathcal{D}_X = \{A \subseteq X \times X \mid \Delta_X \subseteq A \text{ and } A = -A\}$. For $V \in \mathcal{D}_X$ and $x, y \in X$, we define the notation $|x-y| < V$ if $(x,y) \in V$. With this in mind, for $x \in X$ and $V \in \mathcal{D}_X$, we set $B(x,V) = \{y \in X \mid |x-y| < V\}$, the V-ball around x. A *uniformity* $\mathcal{U} \subseteq \mathcal{D}_X$ is a family that satisfies the following:

Exercise 8.8. Prove that if $|x - y| < V$ and $|y-z| < W$ then $|x-z| < V + W$. Perhaps this justifies some of my kooky notation...

(a) If $V \in \mathcal{U}$ and $W \supseteq V$, then $W \in \mathcal{U}$.

(b) If $V, W \in \mathcal{U}$, then $V \cap W \in \mathcal{U}$.

(c) For every $V \in \mathcal{U}$, there exists a $W \in \mathcal{U}$ such that $2W \subseteq V$.

(d) $\bigcap \mathcal{U} = \Delta_X$.

Finally, a family $\mathcal{B} \subseteq \mathcal{U}$ is called a *base* for the uniformity if $\mathcal{B}$ satisfies the following:

(a) For any $V_0, V_1 \in \mathcal{B}$, there exists some $W \in \mathcal{B}$ such that $W \subseteq V_0 \cap V_1$.

(b) For every $V \in \mathcal{B}$ there exists a $W \in \mathcal{B}$ such that $2W \subseteq \mathcal{B}$.

(c) $\bigcap \mathcal{B} = \Delta$.

If X is a set and $\mathcal{U}$ is a uniformity on X, we say that $(X, \mathcal{U})$ is an *uniform space*.

Exercise 8.9. Let $(X, \mathcal{U})$ be a uniform space. Show the set

$$\tau = \{O \subseteq X \mid \text{ for all } x \in O \text{ there exists } V \in \mathcal{U} \text{ such that } B(x,V) \subseteq O\}$$

is a topology on X. We call τ the *topology induced by the uniformity* $\mathcal{U}$.

Examples 8.10. We give some examples of uniformities here.

(a) Let (X,d) be a metric space. Define $U_\varepsilon = \{(x,y) \in X \times X \mid |x-y| < \varepsilon\}$. Then the set

$$\mathcal{U}_d = \{U \subseteq X \times X \mid \text{there exists } \varepsilon > 0 \text{ such that } U_\varepsilon \subseteq U\}$$

is a uniformity on X. This is called the uniformity induced by d.

(b) Let G be a topological group. Let $U \in \mathcal{N}_e$ satisfy $U = U^{-1}$ and define $V_U^R = \{(x,y) \in G \times G \mid xy^{-1} \in U\}$ and $V_U^L = \{(x,y) \in G \times G \mid x^{-1}y \in U\}$. Then we can define

$$\mathcal{U}_L = \left\{V \subseteq G \times G \;\middle|\; \text{there exists } U \in \mathcal{N}(e), U = U^{-1} \text{ such that } V_U^L \subseteq V\right\},$$

$$\mathcal{U}_R = \left\{V \subseteq G \times G \;\middle|\; \text{there exists } U \in \mathcal{N}(e), U = U^{-1} \text{ such that } V_U^R \subseteq V\right\}.$$

Each of these define an (in general different) uniformity on G.

Definition 8.11. Suppose $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces. We say $f : X \to Y$ is *uniformly continuous* if for all $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that whenever $|x - y| < U$ we have that $|f(x) - f(y)| < V$.

Fact 8.12. *Let $(X, \mathcal{U})$ be a uniform space and suppose $\mathcal{B} \subseteq \mathcal{U}$ is a base for $\mathcal{U}$ and $\mathcal{B}$ is countable. Then $(X, \mathcal{U})$ is metrizable.*

9 Extensions

Theorem 9.1 (Urysohn's Lemma)**.** *Suppose that (X, τ) is a normal topological space and let $A, B \subseteq X$ be closed, nonempty, and disjoint. Then there exists a continuous function $f : X \to [0, 1]$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$.*

There is also a notion of separation called Urysohn. Naturally, that is unrelated to Urysohn's lemma.

To construct a continuous function out of nothing, we need some sort of continuous gradation, like a contour plot. To get this object, we prove a lemma.

Lemma 9.2. *Let $\Delta = \left\{ \frac{k}{2^n} \;\middle|\; n \geq 1 \text{ and } 0 \leq k \leq 2^n \right\}$ be the set of dyadic rationals in $(0, 1)$. There is a family $\{U_r\}_{r \in \Delta}$ of open sets in X such that $A \subseteq U_r \subseteq \overline{U_r} \subseteq U_s \subseteq B^{\mathsf{c}}$ for all $r < s$.*

Proof. By normality, we can find $V, W \in \tau$ such that $A \subseteq V \subseteq W^{\mathsf{c}} \subseteq B$. Let $U_{1/2} = V$. Note that $\overline{U_{1/2}} \subseteq W^{\mathsf{c}} \subseteq B$ since W^{c} is closed. Now to inductively select U_r for $r = (2j + 1)2^{-n}$, note that $\overline{U_{j2^{1-n}}}$ and $\left(U_{(j+1)2^{1-n}}\right)^{\mathsf{c}}$ are disjoint closed sets by construction, so we can find $V, W \in \tau$ such that $\overline{U_{j2^{1-n}}} \subseteq V \subseteq W^{\mathsf{c}} \subseteq U_{(j+1)2^{1-n}}$. Set $U_r = V$. This works. ∎

Proof (Urysohn's Lemma). Define $f : X \to [0, 1]$ by $f(x) = \inf\{r \in \Delta \mid x \in U_r\}$. I leave as an exercise to show that $f^{-1}[(-\infty, a)] = \bigcup\limits_{r<a} U_r$ and $f^{-1}[(a, \infty)] = \bigcup\limits_{r>a} \left(\overline{U_s}\right)^{\mathsf{c}}$, thereby showing continuity. ∎

10 Baire Category

Let's prove something spectacular using Baire category.

Theorem 10.1. *Let $f \in C^\infty([0, 1])$ and suppose for each $x \in [0, 1]$, there is some $n \in \mathbb{N}$ such that $f^{(n)}(x) = 0$. Then f is a polynomial.*

Proof. Let $S_n = \left\{ x \in [0, 1] \;\middle|\; f^{(n)}(x) = 0 \right\}$. We note that S_n is closed as the preimage of $\{0\}$ under $f^{(n)}$.

Claim 1. *Suppose that $A \subseteq [0, 1]$ is an interval and for each $x \in A$ we can find some $U \in \mathcal{N}(x)$ such that $f\restriction_U$ is a polynomial. Then f is a polynomial on A.*

Proof. For $x \in A$, pick $\delta > 0$ and polynomial p_x such that $f\restriction_{(x-\delta, x+\delta) \cap [0,1]} = p_x$. Now let $y \in A$. Set $m = \sup\left\{ z \in A \;\middle|\; f\restriction_{[y,z]} = p_y\restriction_{[y,z]} \right\}$. We note that $m \geq y + \delta_y$, and so $(m-\delta_m, m) \cap [y, \sup A]$ contains a nonempty open set. Additionally,

$$p_y\restriction_{(m-\delta_m,m)\cap[y,\sup A]} = f\restriction_{(m-\delta_m,m)\cap[y,\sup A]} = p_m\restriction_{(m-\delta_m,m)\cap[y,\sup A]}.$$

Polynomials that agree on a nonempty open set must agree everywhere, so we conclude that $p_m = p_y$. On the other hand, $f\restriction_{[m,m+\delta_m)\cap A} = p_m\restriction_{[m,m+\delta_m)\cap A}$ and thus $m = \sup A$ - if not, m would no longer be an upper bound. In particular, $f\restriction_{[y,\sup A]} = p_y\restriction_{[y,\sup A]}$. A similar argument shows that $f\restriction_{[\inf A,y]} = p_y\restriction_{[\inf A,y]}$ and so $f = p_y$ on A as desired. ■

By hypothesis, we have that $[0,1] = \bigcup_n S_n$. Let

$$C = \left\{ x \in [0,1] \;\middle|\; \forall U \in \mathcal{N}(x), f\restriction_U \text{ is not a polynomial} \right\}.$$

We will endeavor to show that C is empty, as then Claim 1 will imply the desired result. Towards a contradiction, suppose otherwise.

Claim 2. *C is closed.*

Proof. Let's show C^c is open. Let $x \in C^c$. Then there is some $U \in \mathcal{N}(x)$ such that $f\restriction_U$ is a polynomial. But then for all $y \in U$, we see that $U \in \mathcal{N}(y)$ and so $y \in C^c$, showing C^c is open as desired. ■

The previous claim tells us C is a complete metric space. Furthermore, we see that $C = C \cap [0,1] = \bigcap_n C \cap S_n$. $C \cap S_n$ is closed in C for each n, so the Baire category theorem tells us there is some $n \in \mathbb{N}$ such that $C \cap S_n$ has nonempty interior, so we can find some nonempty open interval I such that $I \cap C \subseteq S_n$.

Claim 3. *C is perfect.*

Proof. Let $x \in C$, and let $\delta > 0$. We wish to find some $y \in (x-\delta, x+\delta) \cap C \setminus \{x\}$. Towards a contradiction, suppose for every $y \in (x-\delta, x+\delta) \setminus \{x\} = (x-\delta, x) \cup (x, x+\delta)$ we could find some neighborhood U_y of y such that $f\restriction_{U_y}$ is a polynomial. In particular, by Claim 1 there are some polynomials p_0 and p_1 such that $f\restriction_{(x-\delta,x)} = p_0\restriction_{(x-\delta,x)}$ and $f\restriction_{(x,x+\delta)} = p_1\restriction_{(x,x+\delta)}$ since each of these intervals are connected. However, we note $p_1^{(n)}(x) = f^{(n)}(x) = p_0^{(n)}(x)$ for all $n \in \mathbb{N}$, so we must have that $p_0 = p_1$ all along. Thus $x \notin C$, contradicting our original assumption, and so C is perfect as desired. ■

Claim 4. *$I \cap C \subseteq S_m \cap C \neq \varnothing$ for each $m \geq n$.*

Proof. We will prove this by induction. By our previous work, we know that $U \cap C \subseteq S_n$. Now suppose that $U \cap C \subseteq S_m$ for some $m \geq n$. Then let $x \in I \cap C$. By Claim 3, we can find some sequence $(x_k)_k$ in $I \cap C \setminus \{x\}$ such that $x_k \to x$. Since $x_k \in I \cap C$, we see that $\frac{f^{(m)}(x_k) - f^{(m)}(x)}{x_k - x} = \frac{0}{x_k - x} = 0$ for all $k \in \mathbb{N}$. Because $f^{(m+1)}(x)$ exists, we can take a limit as $k \to \infty$ to find that $f^{(m+1)}(x) = 0$, and so $x \in S_m$ as desired. ■

Claim 5. $f^{(n)} = 0$ *on* I.

Proof. We know that $f^{(n)} = 0$ on $I \cap C$, so it suffices to show that $f^{(n)} = 0$ on $I \setminus C$. If $I \setminus C$ is empty, we're done, so assume $I \setminus C$ is nonempty. We know $I \setminus C$ is open, so we can write it as a disjoint union of nonempty open intervals $I \setminus C = \bigcup_k I_k$. For each k, we note that $I_k \subseteq C^{\mathsf{c}}$ and is connected, so we can find some polynomial p of degree j such that $p\restriction_{I_k} = f\restriction_{I_k}$. Thus $f^{(m)}\restriction_{I_k}$ is some nonzero constant A. We note since I is an interval and C is closed that $I_k' \cap C \neq \varnothing$, so let $c \in I_k' \cap C$. Then by continuity of $f^{(j)}$, we must have that $f^{(j)}(c) = A$. However, since $c \in C \cap I$, we have that $f^{(m)}(c) = 0$ for all $m \geq n$ by Claim 4, thus we must have that $j < n$. This holds for each k, so we conclude that $f^{(n)} = 0$ on $I \setminus C$ as desired. ■

The previous claim implies that$f$ is a polynomial on I and so $I \subseteq C^{\mathsf{c}}$, contradicting that $I \cap C \neq \varnothing$. Thus C must be empty. This finishes the proof. ■