On a class of anisotropic Sobolev spaces, and an extension of an elementary integral

Subhasish Mukherjee

Department of Mathematics

Carnegie Mellon University

Pittsburgh, PA

A Master's Thesis

Contents

0 Introduction **2**

I A class of anisotropic Sobolev spaces **6**

1 Introduction **7**

1.1 Setup and background 7

1.2 Anisotropic reduction 8

1.3 Main results 9

2 Preliminary estimates **10**

3 When is $X^s_{r,\delta}$ an algebra? **17**

3.1 Splitting E_1 further 21

4 PDE applications **27**

II Extending the Cauchy integral **31**

5 Uniform Function Spaces **31**

5.1 Definitions and Preliminaries 31

5.2 Properties of the uniformity of uniform convergence 31

6 Regulated Functions **33**

6.1 Properties of Regulated Functions 35

6.2 Regulated functions on product spaces 38

6.3 Embeddings of Continuous Functions 41

7 The Cauchy Integral **42**

8 Properties of the Cauchy Integral **44**

8.1 Basic Properties 44

8.2 The Dual of $\mathcal{R}_{\mathscr{A}}(X;\mathbb{R})$ 45

8.3 Convexity of the Integral . . . 46
8.4 Fubini's Theorem . . . 48
8.5 Change of Variables . . . 49
8.6 Fundamental Theorem of Calculus . . . 52

III Appendices **54**

A Some facts about uniformities **54**

A.1 Examples . . . 57

B Some facts about charges **58**

B.1 Peano-Jordan Completion . . . 58
B.2 The Space of Charges . . . 60

0 Introduction

In this thesis, we explore two different topics.

To start, we will embark on an investigation of some special function spaces. When studying PDE, one of the most useful tools we have is that of Sobolev spaces. The classical L^2-based Sobolev norm is defined as $\|f\|_{H^s}^2 = \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} \left|\hat{f}(\xi)\right|^2 \,\mathrm{d}\xi$ with the parameter $s \in \mathbb{R}$. We then define the Sobolev space H^s as

$$H^s(\mathbb{R}^d) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d) \;\middle|\; \hat{f} \in L^1_{\mathrm{loc}} \text{ and } \|f\|_{H^s} < \infty \right\}.$$

These often come up as natural container spaces when working with differential equations.

When working on constructing solutions to the free boundary incompressible Navier-Stokes equations, Leoni and Tice (CITE) discovered that modified versions of these classical spaces were exactly what was needed. They defined the norm

$$\|f\|_{X^s}^2 = \int_{B(0,1)} \frac{|\xi_1|^2 + |\xi|^4}{|\xi|^2} \left|\hat{f}(\xi)\right|^2 \,\mathrm{d}\xi + \int_{B(0,1)^c} |\xi|^{2s} \left|\hat{f}(\xi)\right|^2 \,\mathrm{d}\xi$$

and then the space $X^s(\mathbb{R}^d) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d) \;\middle|\; \hat{f} \in L^1_{\mathrm{loc}} \text{ and } \|f\|_{X^s} < \infty \right\}$. These spaces were unique because they were *anisotropic* - rotations of functions in X^s did not necessarily stay in X^s, as suggested by the dependence on the e_1 direction in the X^s norm. Leoni and Tice then proved several key properties about these spaces, such as completeness and embedding results, which allowed them to use these spaces for PDE. Later, Koganemaru and Tice [5] proved further that X^s was an algebra - that is $f \in X^s$ and $g \in X^s$ then implies $fg \in X^s$, by adapting techniques from the work of Guo, Pausader, and Widmayer [3]. This is an essential property to allow for analytic pertubations of solutions in these spaces. The key technique involved in Koganemaru and Tice's argument was that of Littlewood-Paley theory. This is a way to decompose and localize a function based on its values in the Fourier side, usually based on dyadic annuli. Koganemaru and Tice instead decomposed using conveniently crafted rectangles which helped uncover the fine properties of these spaces.

In the present work, we generalize the X^s spaces by adding two more parameters. We define the norm

$$\|f\|_{X^s_{r,\delta}}^2 = \int_{B(0,1)} \frac{|\xi_1|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \left|\hat{f}(\xi)\right|^2 \, d\xi + \int_{B(0,1)^c} |\xi|^{2s} \left|\hat{f}(\xi)\right|^2 \, d\xi$$

and then the space $X^s_{r,\delta} = \left\{ f \in \mathscr{S}'(\mathbb{R}^d) \;\middle|\; \hat{f} \in L^1_{\mathrm{loc}} \text{ and } \|f\|_{X^s_{r,\delta}} < \infty \right\}$ as before. Leoni and Tice analyzed the case when $\delta = 2$ and $r = 1$. We show these spaces are genuinely anisotropic when $\delta > 1$ and characterize under which parameter regimes the spaces are complete, when H^s embeds into $X^s_{r,\delta}$, and most importantly, when $X^s_{r,\delta}$ is an algebra. We use a modification of the Littlewood-Paley argument used by Koganemaru and Tice. The use of rectangles with a certain aspect ratio allowed us to localize to coordinates where $|\xi_1|^2$ is comparable to $|\xi|^{2\delta}$, granting us the path to properly analyze the low frequency multiplier.

Figures 1 and 2 summarize our findings on when $X^s_{r,\delta}(\mathbb{R}^d)$ was a Banach algebra with the desired embedding properties.

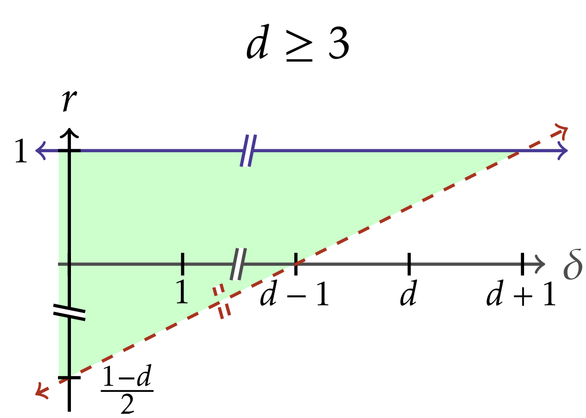

Figure 1: We show that $X^s_{r,\delta}$ is complete when $d > 1 + \delta - 2r$, the region bounded by the red line. The classical Sobolev space H^s embeds into $X^s_{r,\delta}$ when $r \leq 1$, shown by the blue line. In this region, we show that $X^s_{r,\delta}$ is always an algebra.

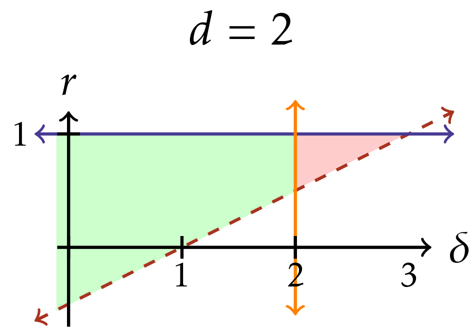

Figure 2: We have the same constraints as in Figure 1 for $d = 2$. However, in this case $X^s_{r,\delta}$ is an algebra if and only if when $\delta \leq 2$, shown by the green region.

In Part II, our work was also guided by analysis of function spaces. The starting point was the classic Cauchy integral, first introduced in the modern form by Bourbaki. The setup is as follows: given a function f from an interval $[a,b]$ to some Banach space Z, we say that f is a step function if f takes on finitely many values and $f^{-1}[\{z\}]$ is a finite union of intervals for every $z \in Z$ - we denote the set of such step functions as $\mathcal{S}([a,b];Z)$. Given a step function $f \in \mathcal{S}([a,b];Z)$, we can define the integral $I(f) \in Z$ in an obvious way - namely, if

$$f = \sum_{k=1}^{n} z_k \chi_{I_k}$$

where each $I_k \subseteq [a,b]$ is a (possibly singleton) interval and each $z_k \in Z$, we set $I(f) = \sum_{k=1}^{n} |I_k| z_k$, where I_k is the size of the interval. One can then show that I defined a bounded linear operator from the step functions (endowed with the uniform topology) into Z, and so admits a natural continuous linear extension on the uniform closure of the step functions, the so-called *regulated* functions, which we denote as $\mathcal{R}([a,b];Z)$. The techniques involved in analysis with the Cauchy integral are all quite elementary, and it presents an alternative introduction to integration. It was argued voraciously by Dieudonné that the Cauchy integral would be a much more pedagogically suitable alternative to a first pass at integration, as compared to the Riemann integral. In the introduction of *Foundations of Modern Analysis*, he writes:

> Only the stubborn conservatism of academic tradition could freeze [the Riemann integral] into a regular part of the curriculum, long after it had outlived its historical importance. Of course, it is perfectly feasible to limit the integration process to a category of functions which is large enough for all purposes of elementary analysis, but close enough to the continuous functions to dispense with any consideration drawn from measure theory; this is what we have done by defining only the integral of regulated functions (sometimes called the "Cauchy integral"). When one needs a more powerful tool, there is no point in stopping halfway, and the general theory of Lebesgue integration is the only sensible answer.

In keeping with the goal of an introduction to integration, we generalize the Cauchy integral to give it more power while still keeping the techniques involved perfectly elementary.

The first generalization comes in the domain, where we look at *charge spaces*. These are sets equipped with an algebra of sets as well as a finitely additive measure (or charge). We can view the interval $[a,b]$ as a charge space in a natural way by restricting to the algebra consisting of finite unions of intervals and the charge induced from the 1-dimensional Lebesgue measure. Indeed, as we will see, the Cauchy integral construction will work on any finite charge space. For those looking to continue studying mathematics, this perspective gives an elementary introdution to the ideas underlying measure theory, and makes the transition to the Lebesgue integral more pleasant. However, if one just wishes to compute integrals, one can do away entirely with the abstraction of generic charge spaces, and just define the appropriate charges on $\mathbb{R}^n$.

The second generalization comes for the codomain. Once we have restricted our domain to be some set X equipped with an algebra $\mathscr{A}$, step functions always make sense regardless of the structure of the codomain. Indeed, in analogy to before, we can set

$$\mathcal{S}_{\mathscr{A}}(X;Z) = \left\{ f : X \to Z \;\middle|\; f[X] \text{ is finite and } f^{-1}[\{z\}] \in \mathscr{A} \text{ for all } z \in Z \right\}$$

In order to take uniform closures, we need the structure of a uniformity. Thus we can define the $\mathscr{A}$-regulated functions as the uniform closure $\mathcal{R}_{\mathscr{A}}(X;Z) = \overline{\mathcal{S}_{\mathscr{A}}(X;Z)}$ for any set X and any uniformity Z.

Every topological vector space Z carries a natural uniformity, so given a charge space $(X, \mathscr{A}, \mu)$ it becomes perfectly natural to define the integral operator $I : \mathcal{S}_{\mathscr{A}}(X;Z) \to Z$ in the same way as before. As we will see, this operator is continuous if and only if Z is locally convex. In this case we can extend I to a continuous linear operator on $\mathcal{R}_{\mathscr{A}}(X;Z)$, giving us the setting for our integral. We will prove most of the standard results about our new integral, including a Riesz representation theorem, ordering properties, absolute continuity of the integral, a weak version of Fubini's theorem, the change of variables formula, and the fundamental theorems of calculus.

We will note that few of the ideas and constructions we present in this section are entirely original - rather, we repackage ideas from various sources and put it together in the context of the Cauchy integral.

Part I

A class of anisotropic Sobolev spaces

1 Introduction

1.1 Setup and background

Consider the problem of finding a solution $v : \mathbb{R}^d \times [0,\infty) \to \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ to the equation $\mathrm{d}tv + (-\Delta)^{\delta/2} v = F$ for some $1 < \delta \in \mathbb{R}$. When $\delta = 2$ this is the standard heat equation. Let us further assume that $F : \mathbb{R}^d \to \mathbb{F}$ is in traveling wave form, namely $F(x,t) = f(x - \gamma t e_1)$ for some $f : \mathbb{R}^d \to \mathbb{F}$ and traveling wave speed $\gamma \in \mathbb{R}\backslash\{0\}$. If we make the traveling wave ansatz $v(x,t) = u(x - \gamma t e_1)$, then we reduce to the PDE $-\gamma \partial_1 u + (-\Delta)^{\delta/2} u = f$ in $\mathbb{R}^d$, which rewrites on the Fourier side as

$$[-2\pi i \gamma \xi_1 + (2\pi|\xi|)^\delta] \hat{u}(\xi) = \hat{f}(\xi). \tag{1.1}$$

Clearly, this determines $\hat{u}$ in terms of $\hat{f}$, and if we assume that $f \in H^s(\mathbb{R}^d;\mathbb{F})$, then we have the estimate

$$\int_{\mathbb{R}^d} (|\xi_1|^2 + |\xi|^{2\delta}) \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 \mathrm{d}\xi \asymp \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 \mathrm{d}\xi = \|f\|^2_{H^s}. \tag{1.2}$$

One can show (and we will do so later) that the space defined by the square-norm on the left is complete and consists of locally integrable functions if and only if $d > 1 + \delta$. Thus, in small dimension it is natural to seek a refinement of this estimate (which requires more information on f, of course) that overcomes this issue and leads to an isomorphism of Banach spaces for the operator $-\gamma \partial_1 + (-\Delta)^{\delta/2}$.

To this end, we write $\mathscr{S}(\mathbb{R}^d;\mathbb{F})$ for the Schwartz space of $\mathbb{F}-$valued functions and $\mathscr{S}'(\mathbb{R}^d;\mathbb{F})$ for the corresponding space of $\mathbb{F}-$valued tempered distributions. Given the parameters $s, r, \delta \in \mathbb{R}$ we define the measurable function $\omega_{s,r,\delta} \colon \mathbb{R}^d \to [0,\infty)$ via

$$\omega_{s,r,\delta}(\xi) = \frac{|\xi_1|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \chi_{B(0,1)}(\xi) + \langle \xi \rangle^{2s} \chi_{B(0,1)^c}(\xi), \tag{1.3}$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ is the usual bracket notation. We then define the Sobolev-type space

$$X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d;\mathbb{F}) \,\middle|\, \hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{C}) \text{ and } \|f\|_{X^s_{r,\delta}} < \infty \right\}, \tag{1.4}$$

where $\hat{\cdot}$ denotes the Fourier transform, and the norm is defined by

$$\|f\|^2_{X^s_{r,\delta}} = \int_{\mathbb{R}^d} \omega_{s,r,\delta}(\xi)\left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi = \int_{B(0,1)} \frac{\xi_1^2 + |\xi|^{2\delta}}{|\xi|^{2r}}\left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi + \int_{B(0,1)^c} \langle \xi \rangle^{2s}\left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi. \tag{1.5}$$

The norm is clearly derived from the associated inner-product

$$\langle f, g \rangle_{X^s_{r,\delta}} = \int_{\mathbb{R}^d} \omega_{s,r,\delta}(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi. \tag{1.6}$$

Note that since $\omega_{s,r,\delta}$ is even, this inner-product takes values in $\mathbb{R}$ when $\mathbb{F} = \mathbb{R}$. We further note that with this notation established, the left side of (1.2) is equivalent to $\|u\|^2_{X^{s+\delta}_{0,\delta}}$.

Although we have motivated the introduction of $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ with a simple linear pseudodifferential equation above, similar issues arose in recent work of the second author and collaborators on the construction of traveling wave solutions to the free boundary Navier-Stokes [6, 10, 5] and Muskat systems [8]. In these instances, the space $X^s_{1,2}(\mathbb{R}^d;\mathbb{F})$ played an essential role in the construction of solutions, and we expect the new more general scale to be useful in other PDE applications. In particular, for uses in nonlinear PDE, the question of when $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is an algebra is of central importance.

1.2 Anisotropic reduction

Consider the case $\delta \leq 1$. Then for $\xi \in \mathbb{R}^d$ such that $|\xi| \leq 1$ we have that

$$\omega_{s,r,\delta}(\xi) = \frac{|\xi_1|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \asymp \frac{|\xi|^{2\delta}}{|\xi|^{2r}} = |\xi|^{2(\delta - r)}, \tag{1.7}$$

and so $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}) = \dot{H}^{(\delta-r,s)}(\mathbb{R}^d;\mathbb{F})$, where for $\lambda, \rho \in \mathbb{R}$ we define the bihomogeneous Sobolev space

$$\dot{H}^{(\lambda,\rho)}(\mathbb{R}^d;\mathbb{F}) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d;\mathbb{F}) \;\middle|\; \hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{C}) \text{ and } \|f\|_{\dot{H}^{(\lambda,\rho)}} < \infty \right\} \tag{1.8}$$

with

$$\|f\|^2_{\dot{H}^{(\lambda,\rho)}} = \int_{B(0,1)} |\xi|^{2\lambda}\left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi + \int_{B(0,1)^c} |\xi|^{2\rho}\left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi. \tag{1.9}$$

This shows that when $\delta \leq 1$ the space $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is actually isotropic, and the pair of parameters (r, δ) reduce to the single parameter $\delta - r \in \mathbb{R}$.

Similarly, consider the case $d = 1$. We then note that for $\xi \in \mathbb{R}$ with $|\xi| < 1$ we have

$$\omega_{s,r,\delta}(\xi) = \frac{|\xi_1|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} = \frac{|\xi|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \asymp |\xi|^{\min\{2(\delta-r), 2(1-r)\}} \tag{1.10}$$

and so again we reduce to $X^s_{r,\delta}(\mathbb{R};\mathbb{F}) = \dot{H}^{\delta-r,s}(\mathbb{R};\mathbb{F})$ or $X^s_{r,\delta}(\mathbb{R};\mathbb{F}) = \dot{H}^{1-r,s}(\mathbb{R};\mathbb{F})$ depending on whether $\delta \leq 1$ or $\delta > 1$.

As such, in this paper we will focus our attention on the more interesting regime $\delta > 1$ and $d \geq 2$, in which case the space $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is genuinely anisotropic, as we will see later.

1.3 Main results

Our goal in the present paper is two-fold. First, we aim to study the functional analytic properties of this generalized scale, including embeddings into classical spaces, completeness, and under which parameter regime this space is an algebra. Second, we will provide some simple uses of these spaces in constructing traveling wave solutions to some simple PDEs to provide an elementary demonstration of the use of this type of space.

The following theorem summarizes the properties of $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ we will prove in Sections 2 and 3. Then in Section 4 we will record the PDE applications.

Theorem 1.1. Let $s, r, \delta \in \mathbb{R}$ and $d \in \mathbb{N}$ with $\delta > 1$ and $d \geq 2$. Then the following hold.

(a) $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is a Hilbert space if and only if $1 + \delta - 2r < d$. In either case, we have the continuous inclusion $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}) \hookrightarrow C^\infty_0(\mathbb{R}^d;\mathbb{F}) + H^s(\mathbb{R}^d;\mathbb{F})$, where $C^\infty_0(\mathbb{R}^d;\mathbb{F}) = \bigcap_{k\in\mathbb{N}} C^k_0(\mathbb{R}^d;\mathbb{F})$ is endowed with its standard Fréchet topology.

(b) $H^s(\mathbb{R}^d;\mathbb{F}) \hookrightarrow X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ if and only if $r \leq 1$.

(c) If $1 + \delta - 2r < d$ and $r \leq 1$ then $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is anisotropic in the sense that it is not closed under composition with rotations. More precisely, there exist $f \in X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}) \cap C^\infty_0(\mathbb{R}^d;\mathbb{F})$ such that $f \circ Q \notin X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ whenever $Q \in O(d)$ satisfies $|Qe_1 \cdot e_1| < 1$. In particular, the subspace inclusion $H^s(\mathbb{R}^d;\mathbb{F}) \subset X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is strict in this parameter regime.

(d) If $d > 1 + \delta - 2r$ and $s > d/2$, $f \in X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ and $g \in H^s(\mathbb{R}^d;F)$, then $fg \in H^s(\mathbb{R}^d;\mathbb{F})$ and there exists a constant $C > 0$ such that $\|fg\|_{H^s} \leq C\|f\|_{X^s_{r,\delta}}\|g\|_{H^s}$

(e) Suppose $d > 1 + \delta - 2r$, $r \leq 1$, and $s > d/2$. If $d \geq 3$, then $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is an algebra. If $d = 2$, then $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is an algebra if and only if $\delta \leq 2$. In particular, in this parameter regime, when $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is an algebra, (4) says that $H^s(\mathbb{R}^d;\mathbb{F}) \subset X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is an ideal.

(f) If $\delta - r \leq t \leq s$ and $f \in X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$, then $(-\Delta)^{t/2} f \in H^{s-t}(\mathbb{R}^d;\mathbb{F})$ and $\partial_1 f \in \dot{H}^{-r}(\mathbb{R}^d;\mathbb{F})$ and $\left\|(-\Delta)^{t/2} f\right\|_{H^{s-t}} + \|\partial_1 f\|_{\dot{H}^{-r}} \lesssim \|f\|_{X^s_{r,\delta}}$.

The regions where $X^s_{r,\delta}$ is an algebra are outlined in Figure 3.

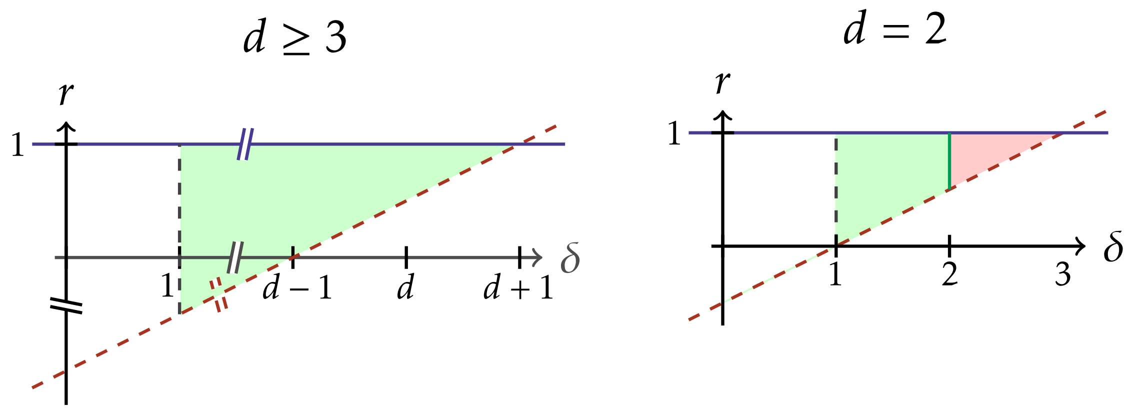

Figure 3: On the left: For $d \geq 3$, we restrict to the anisotropic region with the grey line marking $\delta > 1$. We show that $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is complete when $d > 1 + \delta - 2r$, the region bounded by the red line. The classical Sobolev space H^s embeds into $X^s_{r,\delta}$ when $r \leq 1$, shown by the blue line. In this region, we show that $X^s_{r,\delta}$ is always an algebra. On the right: We have the same constraints for $d = 2$. However, in this case $X^s_{r,\delta}$ is an algebra if and only if when $\delta \leq 2$, marked by the green line and everything to its left.

2 Preliminary estimates

We begin with some useful estimates and bounds. To motivate the first we make the following remark.

Remark 2.1. The unit radius employed in $\omega_{s,r,\delta}$ is not essential. Indeed, it's straightforward to verify that the map

$$X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}) \ni f \mapsto \left(\int_{B(0,R)} \frac{\xi_1^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi + \int_{B(0,R)^c} \langle \xi \rangle^{2s} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi \right)^{1/2} \tag{2.1}$$

yields an equivalent norm for every $R > 0$.

The next result is a crucial property characterizing integrability of the reciprocal of our multiplier $\omega_{s,r,\delta}$ around the origin.

Lemma 2.2. Let $1 \leq d \in \mathbb{N}$ and suppose that $r, \delta \in \mathbb{R}$ with $d > 1 + \delta - 2r$ and $\delta > 1$. If $R > 0$, then

$$\int_{B(0,R)} \frac{|\xi|^{2r}}{\xi_1^2 + |\xi|^{2\delta}} \, \mathrm{d}\xi < \infty. \tag{2.2}$$

Proof. We first consider the case when $d = 2$, in which case we compute:

$$\int_{B(0,R)} \frac{|\xi|^{2r}}{\xi_1^2 + |\xi|^{2\delta}} \,\mathrm{d}\xi = \int_0^R \int_0^{2\pi} \frac{\rho^{2r}}{\rho^2 \cos^2(\theta) + \rho^{2\delta}} \rho \,\mathrm{d}\theta \mathrm{d}\rho$$
$$= \int_0^R \rho^{2r-1} \int_0^{2\pi} \frac{1}{\cos^2(\theta) + \rho^{2\delta-2}} \,\mathrm{d}\theta \mathrm{d}\rho = 2\pi \int_0^R \rho^{2r-1} \frac{1}{\rho^{\delta-1}\sqrt{1+\rho^{2\delta-2}}} \,\mathrm{d}\rho \quad (2.3)$$

Since $\delta > 1$, we know $\sqrt{1+\rho^{2\delta-2}} \asymp 1$ in $B(0,R)$ and so the last integral is finite if and only if $2r - \delta > -1 = 1 - d$.

Next suppose $d \geq 3$. We can write (2.2) in spherical coordinates to find

$$I := \int_{B(0,R)} \frac{|\xi|^{2r}}{\xi_1^2 + |\xi|^{2\delta}} \,\mathrm{d}\xi = \int_0^R \int_0^{2\pi} \int_{[0,\pi]^{d-2}} \frac{\rho^{2r}}{(\rho \cos \varphi_{d-2})^2 + \rho^{2\delta}} \rho^{d-1} (\sin \varphi_{d-2})^{d-2} g_d(\varphi_1, \ldots, \varphi_{d-3}) \,\mathrm{d}\phi \mathrm{d}\theta \mathrm{d}\rho \quad (2.4)$$

where $\mathrm{d}\phi = \prod_{i=1}^{d-2} \mathrm{d}\varphi_i$ and $g_d(\varphi_1, \ldots, \varphi_{d-3}) = \prod_{i=1}^{d-3} \sin^i(\phi_i)$. Integrating over $\varphi_1, \ldots, \varphi_{d-3}$ and changing variables with $u = \sin \phi_{d-2}$ we have

$$I = C(d) \int_0^R \int_0^\pi \frac{\rho^{2r+d-1}}{\rho^2 \cos^2 \varphi_{d-2} + \rho^{2\delta}} (\sin \varphi_{d-2})^{d-2} \,\mathrm{d}\varphi_{d-2} \mathrm{d}\rho = C(d) \int_0^R \int_{-1}^1 \frac{\rho^{2r+d-1}}{\rho^2 u^2 + \rho^{2\delta}} (1-u^2)^{\frac{d-3}{2}} \,\mathrm{d}u \mathrm{d}\rho$$
$$\leq C(d) \int_0^R \rho^{2r+d-3} \int_{-1}^1 \frac{1}{u^2 + \rho^{2\delta-2}} \mathrm{d}u \mathrm{d}\rho = 2C(d) \int_0^R \rho^{2r+d-3} \frac{1}{\rho^{\delta-1}} \arctan\left(\frac{1}{\rho^{\delta-1}}\right) \mathrm{d}\rho$$
$$\asymp \int_0^R \rho^{2r+d-\delta-2} \,\mathrm{d}\rho, \quad (2.5)$$

where we used that $\arctan(\rho^{1-\delta}) \asymp 1$ for $\rho < R$ since $\delta > 1$. The latter integral is finite if and only if $d > 1 + \delta - 2r$, giving the desired result. ■

Using the previous lemma, we can now discern when $\|\hat{f}\|_{L^1}$ is bounded by $\|f\|_{X^s_{r,\delta}}$ and show that functions in $X^s_{r,\delta}$ are sums of a smooth function and something in H^s.

Proposition 2.3. Suppose that $d > 1 + \delta - 2r$, and let $R > 0$. Then the following hold.

(a) There exists a constant $C = C(R, r, d, \delta, s) > 0$ such that if $f \in X^s_{r,\delta}(\mathbb{R}^d; \mathbb{F})$, then

$$\int_{B(0,R)} \left|\hat{f}(\xi)\right| \mathrm{d}\xi + \left(\int_{B(0,R)^c} (1 + |\xi|^2)^s \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{1/2} \leq C \|f\|_{X^s_{r,\delta}}. \quad (2.6)$$

Moreover, if $s > d/2$, then $\left\|\hat{f}\right\|_{L^1} \leq C \|f\|_{X^s_{r,\delta}}$ for some $C = C(r, d, \delta, s) > 0$.

(b) For $f \in X^s_{r,\delta}(\mathbb{R}^d; \mathbb{F})$ define $f_{l,R} = (\hat{f} \chi_{B(0,R)})^\vee$ and $f_{h,R} = (\hat{f} \chi_{B(0,R)^c})^\vee$. Then $f_{l,R}, f_{h,R} \in X^s_{r,\delta}(\mathbb{R}^d; \mathbb{F})$, $f = f_{l,R} + f_{h,R}$, and we have the bounds $\|f_{l,R}\|_{X^s_{r,\delta}} \leq \|f\|_{X^s_{r,\delta}}$ and $\|f_{h,R}\|_{X^s_{r,\delta}} \leq \|f\|_{X^s_{r,\delta}}$. Moreover, for each $k \in \mathbb{N}$

we have that $f_{l,R} \in C_b^k(\mathbb{R}^d;\mathbb{F})$ with the estimate $\|f_{l,R}\|_{C_b^k} \leq C(k)\|f_{l,R}\|_{X^s_{r,\delta}}$, and $f_{h,R} H^s(\mathbb{R}^d;\mathbb{F})$ with the estimate $\|f_{h,R}\|_{H^s} \lesssim \|f_{h,R}\|_{X^s_{r,\delta}}$.

(c) We have the continuous inclusion $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}) \hookrightarrow C_0^\infty(\mathbb{R}^d;\mathbb{F}) + H^s(\mathbb{R}^d;\mathbb{F})$.

Proof. We begin with the proof of the first item. Clearly, the second term on the left side of (2.6) is bounded by the term on the right, so we only need to consider the first. We estimate the first term using Cauchy-Schwarz, Lemma 2.2, and Remark 2.1:

$$\int_{B(0,R)} \left|\hat{f}(\xi)\right| \mathrm{d}\xi \leq \left(\int_{B(0,R)} \frac{|\xi|^{2r}}{\xi_1^2 + |\xi|^{2\delta}} \mathrm{d}\xi\right)^{1/2} \left(\int_{B(0,R)} \frac{\xi_1^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{1/2} \leq C\|f\|_{X^s_{r,\delta}}. \tag{2.7}$$

Additionally, if $s > d/2$, we can estimate

$$\int_{B(0,R)^c} \left|\hat{f}(\xi)\right| \mathrm{d}\xi \leq \left(\int_{B(0,R)^c} \frac{1}{(1+|\xi|^2)^s} \mathrm{d}\xi\right)^{1/2} \left(\int_{B(0,R)^c} \left(1+|\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{1/2} \leq C\|f\|_{X^s_{r,\delta}}. \tag{2.8}$$

Combining this with (2.6) gives the second inequality and completes the proof of the first item. The second and third items follows easily from the first and the standard properties of band-limited functions whose Fourier transforms are in L^1. ■

With this lemma in hand, we can now characterize when $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is complete.

Theorem 2.4. $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is a Hilbert space if and only if $1+\delta-2r < d$.

Proof. First suppose that $r > \frac{1+\delta-d}{2}$ and that $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in $X^s_{r,\delta}$. Write μ for the measure $\omega_{s,r,\delta}(\xi)\mathrm{d}\xi$ on $\mathbb{R}^d$. Then $\{\hat{f}_k\}_{k\in\mathbb{N}}$ is Cauchy in $L^2_\mu(\mathbb{R}^d;\mathbb{F})$, and so there exists some $F \in L^2_\mu(\mathbb{R}^d;\mathbb{F})$ such that $\hat{f}_k \to F$ in $L^2_\mu(\mathbb{R}^d;\mathbb{F})$ as $k \to \infty$. We now aim to verify that $F \in \mathscr{S}' \cap L^1_{\mathrm{loc}}$. Employing Cauchy-Schwarz and Lemma 2.2, we have that

$$\int_{B(0,1)} |F| \leq \left(\int_{B(0,1)} \frac{1}{\omega_{s,r,\delta}}\right)^{1/2} \left(\int_{B(0,1)} |F|^2 \omega_{s,r,\delta}\right)^{1/2} \lesssim \|F\|_{L^2_\mu} < \infty, \tag{2.9}$$

or in other words $F\chi_{B(0,R)} \in L^1$. Since $\omega_{s,r,\delta}(\xi) = \langle\xi\rangle^{2s}$ for $|\xi| \geq 1$, it's also clear that $F\chi_{B(0,R)^c} \in \mathscr{S}' \cap L^1_{\mathrm{loc}}$. Hence, $F = F\chi_{B(0,R)} + F\chi_{B(0,R)^c} \in\in \mathscr{S}' \cap L^1_{\mathrm{loc}}$, and so we may define $f = \check{F} \in \mathscr{S}'(\mathbb{R}^d;\mathbb{F})$. It's then clear that $f \in X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ and $f_k \to f$ in $X^s_{r,\delta}$, which shows that $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is complete.

Conversely, suppose $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is complete when $\delta \geq 1$. Consider the norm $\|\cdot\|_* : X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}) \to [0,\infty)$ defined by $\|f\|_* = \|\hat{f}\chi_{B(0,1)}\|_{L^1} + \|f\|_{X^s_{r,\delta}}$, which is well-defined thanks to the fact that $\hat{f} \in L^1_{\mathrm{loc}}$ for any $f \in X^s_{r,\delta}$. Note that $(X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}); \|\cdot\|_*)$ is also complete. Then the identity $I : (X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}); \|\cdot\|_*) \to (X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}); \|\cdot\|_{X^s_{r,\delta}})$

is obviously a continuous surjective linear map, so by the open mapping theorem $\|f\|_* \lesssim \|f\|_{X^s_{r,\delta}}$. In particular, we have that $\|\hat{f}\chi_{B(0,1)}\|_{L^1} \lesssim \|f\|_{X^s_{r,\delta}}$.

Now let $0 < \varepsilon < \frac{1}{2}$ and define the rectangle $R_\varepsilon = [\varepsilon^\delta/2, 3\varepsilon^\delta/2] \times [\varepsilon/2, 3\varepsilon/2]^{d-1}$. Let $F = \chi_{R_\varepsilon \cup -R_\varepsilon}$. Then for $\xi \in R_\varepsilon \cup -R_\varepsilon$ we have $|\xi_1| \asymp \varepsilon^\delta$ and $|\xi| \asymp \varepsilon$, and so

$$\omega_{s,r,\delta}(\xi) \asymp \frac{|\xi_1|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \asymp \frac{\varepsilon^{2\delta} + \varepsilon^{2\delta}}{\varepsilon^{2r}} \asymp \varepsilon^{2(\delta-r)}. \tag{2.10}$$

Simple computations show that

$$\|F\|_{L^1} = 2|R_\varepsilon| \asymp \varepsilon^{d-1+\delta} \text{ and } \left\|\check{F}\right\|_{X^s_{r,\delta}} \asymp \sqrt{|R_\varepsilon|\varepsilon^{2(\delta-r)}} \asymp \varepsilon^{(3\delta+d-2r-1)/2}. \tag{2.11}$$

Combining the preceding analysis, we have $\varepsilon^{d-1+\delta} \asymp \|F\|_{L^1} \lesssim \|\check{F}\|_{X^s_{r,\delta}} \asymp \varepsilon^{(3\delta+d-2r-1)/2}$. Sending $\varepsilon \to 0$, this implies $(3\delta + d - 2r - 1)/2 < d - 1 + \delta$ giving us the bound $d > 1 + \delta - 2r$ as desired. ∎

We can also characterize exactly when the Schwartz functions $\mathscr{S}$ and the classical Sobolev space H^s embeds into $X^s_{r,\delta}$. We first prove a useful lemma.

Lemma 2.5. Let $r, \delta \in \mathbb{R}$ with $\delta \geq 1$. Then

$$J = \int_{B(0,1)} \frac{|\xi_1|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \, \mathrm{d}\xi < \infty \tag{2.12}$$

if and only if $r < \frac{2+d}{2}$, where $B(0,1)$ is the unit ball in $\mathbb{R}^d$.

Proof. We write $J_1 = \int_{B(0,1)} \frac{|\xi_1|^2}{|\xi|^{2r}} \, \mathrm{d}\xi$ and $J_2 = \int_{B(0,1)} \frac{|\xi|^{2\delta}}{|\xi|^{2r}} \, \mathrm{d}\xi$, so we have that $J = J_1 + J_2$. We know that J_2 is finite if and only if $2(\delta - r) > -d$, so it only remains to analyze J_1. We first note by symmetry for any $1 \leq k \leq d$ we have that

$$J_1 = \int_{B(0,1)} \frac{|\xi_k|^2}{|\xi|^{2r}} \, \mathrm{d}\xi, \text{ and hence } dJ = \sum_{k=1}^{d} \int_{B(0,1)} \frac{|\xi_k|^2}{|\xi|^2 r} \, \mathrm{d}\xi = \int_{B(0,1)} \frac{|\xi|^2}{|\xi|^{2r}} \, \mathrm{d}\xi. \tag{2.13}$$

The latter is finite if and only if $2 - 2r > -d$. Since $\delta \geq 1$, this is the more restrictive condition, and so we have that $J < \infty$ if and only if $r < \frac{2+d}{2}$. ∎

We now characterize investigate how the Schwartz functions relate to our spaces.

Theorem 2.6. For all $s, r \in \mathbb{R}$ and $\delta \geq 1$, we have $\mathscr{S}(\mathbb{R}^d;\mathbb{F}) \cap X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is dense in $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$. Furthermore, we have that $\mathscr{S}(\mathbb{R}^d;\mathbb{F}) \subseteq X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ if and only if $r < \frac{2+\delta}{2}$.

Proof. Let $f \in X^s_{r,\delta}(\mathbb{R}^d)$ and $\varepsilon > 0$. For $0 < R_1 < R_2 < \infty$, define the annulus $A(R_1, R_2) = B(0, R_2) \smallsetminus B[0, R_1]$. By the monotone convergence theorem, we may find $0 < R_1 < R_2 < \infty$ such that

$$\int_{A(R_1,R_2)^c} \omega_{s,r,\delta}(\xi) \left| \hat{f}(\xi) \right|^2 \, \mathrm{d}\xi < \frac{\varepsilon^2}{4}. \tag{2.14}$$

Pick nonnegative and radially symmetric $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\varphi) \subseteq B(0,1)$ and $\int_{\mathbb{R}^d} \varphi = 1$. Then for $0 < \eta < \frac{R}{4}$, define the function $F_\eta \in C_c^\infty(\mathbb{R}^d)$ by

$$F_\eta(\xi) = \int_{A(R_1,R_2)} \frac{1}{\eta^d} \varphi\left(\frac{\xi - z}{\eta} \right) \hat{f}(z) \, \mathrm{d}z. \tag{2.15}$$

An elementary computation shows that $\operatorname{supp}(F_\eta) \subseteq A(R_1/2, R_1+R_2)$ and $\overline{F_\eta(\xi)} = F_\eta(-\xi)$ which then implies $\check{F}_\eta \in \mathscr{S}(\mathbb{R}^d)$ is real-valued by the same argument as in [6]. On the annulus $A(R_1/2, R_1 + R_2)$, we know that $\omega_{s,r,\delta}(\xi) \asymp 1$, and so the usual theory of mollification (CITE) supplies us with some $0 < \eta_0 < R_1/2$ such that

$$\int_{A(R_1,R_2)} \omega_{s,r,\delta}(\xi) \left| \hat{f}(\xi) - F_{\eta_0}(\xi) \right|^2 \mathrm{d}\xi + \int_{A(R_1/2,R_1+R_2) \smallsetminus A(R_1,R_2)} \omega_{s,r,\delta}(\xi) \left| F_{\eta_0}(\xi) \right|^2 \mathrm{d}\xi < \frac{\varepsilon^2}{8} \tag{2.16}$$

Thus if we define $f_{\eta_0} = \check{F}_{\eta_0}$, then $f_{\eta_0} \in X^s_{r,\delta}(\mathbb{R}^d) \cap \mathscr{S}(\mathbb{R}^d)$ and $\operatorname{supp}\left(\hat{f}_{\eta_0}\right) \subseteq A(R_1/2, R_1 + R_2)$. Putting everything together, we see

$$\begin{aligned}
\left\| f - f_{\eta_0} \right\|^2_{X^s_{r,\delta}} &= \int_{A(R_1/2,R_1+R_2)^c} \omega_{s,r,\delta}(\xi) \left| \hat{f}(\xi) \right|^2 \mathrm{d}\xi + \int_{A(R_1/2,R_1+R_2)} \omega_{s,r,\delta}(\xi) \left| \hat{f}(\xi) - F_{\eta_0}(\xi) \right|^2 \mathrm{d}\xi \\
&< \frac{\varepsilon^2}{4} + \int_{A(R_1,R_2)} \omega_{s,r,\delta}(\xi) \left| \hat{f}(\xi) - F_{\eta_0}(\xi) \right|^2 \mathrm{d}\xi + \int_{A(R_1/2,R_1+R_2) \smallsetminus A(R_1,R_2)} \omega_{s,r,\delta}(\xi) \left| \hat{f}(\xi) - F_{\eta_0}(\xi) \right|^2 \mathrm{d}\xi \\
&< \frac{\varepsilon^2}{4} + \int_{A(R_1,R_2)} \omega_{s,r,\delta}(\xi) \left| \hat{f}(\xi) - F_{\eta_0}(\xi) \right|^2 \mathrm{d}\xi + 2 \int_{A(R_1/2,R_1+R_2) \smallsetminus A(R_1,R_2)} \omega_{s,r,\delta}(\xi) \left| F_{\eta_0}(\xi) \right|^2 \mathrm{d}\xi \\
&\qquad + 2 \int_{A(R_1,R_2)^c} \omega_{s,r,\delta}(\xi) \left| \hat{f}(\xi) \right|^2 \mathrm{d}\xi < \frac{\varepsilon^2}{4} + 2\frac{\varepsilon^2}{8} + 2\frac{\varepsilon^2}{4} < \varepsilon^2.
\end{aligned} \tag{2.17}$$

Since ε was arbitrary, we have shown that $\mathscr{S}(\mathbb{R}^d; \mathbb{F}) \cap X^s_{r,\delta}(\mathbb{R}^d; \mathbb{F})$ is dense in $X^s_{r,\delta}(\mathbb{R}^d; \mathbb{F})$.

For the second assertion, first suppose that $\mathscr{S}(\mathbb{R}^d; \mathbb{F}) \subseteq X^s_{r,\delta}(\mathbb{R}^d; \mathbb{F})$. Then pick some radial $\varphi \in \mathscr{S}(\mathbb{R}^d; \mathbb{R})$ such that $\varphi = 1$ on $B(0,1)$ and $\varphi \geq 0$ everywhere. Then $\check{\varphi} \in X^s_{r,\delta}(\mathbb{R}^d; \mathbb{F})$, so we see that

$$\infty > \|\check{\varphi}\|^2_{X^s_{r,\delta}} \geq \int_{B(0,1)} \frac{|\xi_1|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \, \mathrm{d}\xi. \tag{2.18}$$

By Lemma 2.5, we must have $r < \frac{2+d}{2}$.

Conversely, suppose that $r < \frac{2+d}{2}$. Let $\varphi \in \mathscr{S}(\mathbb{R}^d; \mathbb{F})$ and let $\varphi_0 = \left(\hat{\varphi} \chi_{B(0,1)} \right)^{\vee}$ and $\varphi_1 = \left(\hat{\varphi} \chi_{B(0,1)^c} \right)^{\vee}$. We then see

$$\|\varphi_0\|^2_{X^s_{r,\delta}} = \int_{B(0,1)} |\hat{\varphi}(\xi)|^2 \frac{|\xi_1|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \,\mathrm{d}\xi < \infty \tag{2.19}$$

by Lemma 2.5, and clearly since $\varphi \in \mathscr{S}$ we have

$$\|\varphi_1\|^2_{X^s_{r,\delta}} = \int_{B(0,1)^c} |\hat{\varphi}(\xi)|^2 |\xi|^{2s} \,\mathrm{d}\xi < \infty. \tag{2.20}$$

We know that $\hat{\varphi} \in \mathscr{S} \subseteq L^1_{\mathrm{loc}}$, thus $\varphi \in X^s_{r,\delta}$ as desired. ∎

Next we investigate when standard Sobolev spaces embed into the anisotropic ones.

Theorem 2.7. $H^s(\mathbb{R}^d;\mathbb{F}) \hookrightarrow X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ if and only if $r \le 1$.

Proof. Suppose initially that $r \le 1$. Then for $|\xi| \le 1$ we can use the fact that $\delta > 1$ to bound

$$\omega_{s,r,\delta}(\xi) = \frac{\xi_1^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \lesssim |\xi|^{2-2r} + |\xi|^{2\delta-2r} \lesssim |\xi|^{2-2r} \lesssim 1. \tag{2.21}$$

From this we readily deduce the continuous embedding $H^s(\mathbb{R}^d;\mathbb{F}) \hookrightarrow X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$.

Conversely, suppose we have the continuous embedding $H^s(\mathbb{R}^d;\mathbb{F}) \hookrightarrow X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$, and write $C \ge 0$ for the embedding constant. Restricting to $f \in \mathscr{S}(\mathbb{R}^d;\mathbb{F})$ such that $\mathrm{supp}(\hat{f}) \subset B(0,1)$, we find that

$$\int_{B(0,1)} \frac{\xi_1^2 + |\xi|^{2\delta}}{|\xi|^{2r}} |\hat{f}(\xi)|^2 \mathrm{d}\xi \le C^2 \int_{B(0,1)} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 \mathrm{d}\xi, \tag{2.22}$$

and since this must hold for all such f, we deduce the pointwise bound

$$\frac{\xi_1^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \le C^2 \langle \xi \rangle^{2s} \lesssim 1. \tag{2.23}$$

In turn, this implies that $\xi_1^2 \lesssim \xi_1^{2r}$ for $|\xi_1| < 1$, and hence $r \le 1$. ∎

However, in this parameter regime this space is genuinely bigger than $H^s(\mathbb{R}^d;\mathbb{F})$, and also anisotropic - in general, if $f \in X^s_{r,\delta}$, it is not true that $f \circ Q \notin X^s_{r,\delta}$ for every nonidentity linear isometry Q.

Theorem 2.8. Suppose that $r \le 1$, and $d > \delta - 2r + 1$. Let $Q \in O(d) = \{M \in \mathbb{R}^{d\times d} M^\intercal M = I\}$ be such that $|Qe_1 \cdot e_1| < 1$. Then there exists $f \in X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}) \cap C^\infty_0(\mathbb{R}^d;\mathbb{F}) \setminus H^s(\mathbb{R}^d;\mathbb{F})$ such that $f \circ Q \notin X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$.

Proof. Let $Q \in O(d)$ with $|Qe_1 \cdot e_1| < 1$. For $1 \le k \le d$ set $\sigma_k = 1$ if $Qe_1 \cdot e_k \ge 0$ and $\sigma_k = -1$ if $Qe_1 \cdot e_k < 0$. For $\varepsilon > 0$ set $R_\varepsilon = \sigma_1 \left[\varepsilon^\delta/2, 3\varepsilon^\delta/2\right] \times \prod_{k=2}^d \sigma_k \left[\varepsilon/2, 3\varepsilon/2\right]$. By construction, for each $0 < \varepsilon < \frac{2}{3\sqrt{d}}$ and $\xi \in R_\varepsilon \cup (-R_\varepsilon) \subseteq$

$B(0,1)$ we have that

$$|Q^\intercal\xi\cdot e_1| = \left|\sum_{k=1}^d \xi_k\,(e_k\cdot Qe_1)\right| = \left|\sum_{k=1}^d \sigma_k\xi_k|e_k\cdot Qe_1|\right| = \sum_{k=1}^d |\xi_k||e_k\cdot Qe_1| \asymp \varepsilon^\delta|e_1\cdot Qe_1| + \sum_{k=2}^d |e_k\cdot Qe_1|\varepsilon \asymp \varepsilon, \tag{2.24}$$

where the last equivalence follows because $\delta > 1$ and we can find some $j \geq 2$ with $\left|e_j\cdot Qe_1\right| > 0$. Furthermore, again because $\delta > 1$, we have that $|\xi| \asymp \varepsilon$ for $\xi \in R_\varepsilon \cup (-R_\varepsilon)$. We thus readily deduce the equivalences

$$\omega_{s,r,\delta}(\xi) = \frac{|\xi_1|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \asymp \frac{\varepsilon^{2\delta}+\varepsilon^{2\delta}}{\varepsilon^{2r}} \asymp \varepsilon^{2\delta-2r} \text{ and} \tag{2.25}$$

$$\omega_{s,r,\delta}(Q^\intercal\xi) = \frac{|Q^\intercal\xi\cdot e_1|^2 + |Q^\intercal\xi|^{2\delta}}{|Q^\intercal\xi|^{2r}} = \frac{|Q^\intercal\xi\cdot e_1|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} \asymp \frac{\varepsilon^2+\varepsilon^{2\delta}}{\varepsilon^2 r} \asymp \varepsilon^{2-2r} \tag{2.26}$$

for $\xi \in R_\varepsilon \cup (-R_\varepsilon)$.

Set $F_\varepsilon = \chi_{R_\varepsilon} + \chi_{-R_\varepsilon}$ and note $F_\varepsilon(-\xi) = F_\varepsilon(\xi) = \overline{F_\varepsilon(\xi)}$. The previous calculations then show that

$$\|F_\varepsilon\|^2_{X^s_{r,\delta}} = \int_{\mathbb{R}^d} \omega_{s,r,\delta}(\xi)|F_\varepsilon(\xi)|^2\,\mathrm{d}\xi \asymp \varepsilon^{2\delta-2r}\varepsilon^{\delta+d-1} = \varepsilon^{3\delta-2r+d-1}, \tag{2.27}$$

$$\|F_\varepsilon\|_{L^1} = \|F_\varepsilon\|^2_{L^2} = |R_\varepsilon| + |-R_\varepsilon| \asymp \varepsilon^{\delta+d-1}, \text{ and} \tag{2.28}$$

$$\|F_\varepsilon\circ Q\|^2_{X^s_{r,\delta}} = \int_{\mathbb{R}^d} \omega_{s,r,\delta}(\xi)|F_\varepsilon(Q\xi)|^2\,\mathrm{d}\xi = \int_{\mathbb{R}^d} \omega_{s,r,\delta}(Q^\intercal\xi)|F_\varepsilon(\xi)|^2\,\mathrm{d}\xi \asymp \varepsilon^{2-2r}\varepsilon^{\delta+d-1} = \varepsilon^{1-2r+\delta+d}. \tag{2.29}$$

Let $\alpha = \frac{d+\delta+1-2r}{2}$ and fix $K \in \mathbb{N}$ with $4^K > \frac{3\sqrt{d}}{2}$. Set $F = \sum_{k\geq K} 4^{\alpha k} F_{4^{-k}}$ and note that $\operatorname{supp}(F_{4^{-k}}) \cap \operatorname{supp}(F_{4^{-j}})$ are disjoint for $j,k \geq K$ and $j \neq k$. Now we compute various norms of F. First, using that $d > \delta - 2r + 1$ we see $\alpha - \delta - d + 1 = \frac{3-d-\delta-2r}{2} < 1-\delta < 0$, and so

$$\int_{\mathbb{R}^d} |F(\xi)|\,\mathrm{d}\xi \asymp \sum_{k=K}^\infty 4^{\alpha k}4^{-k(\delta+d-1)} = \sum_{k=K}^\infty 4^{\frac{3d-\delta-1}{2}-r} < \sum_{k=K}^\infty 4^{k(1-\delta)} < \infty. \tag{2.30}$$

The definition of α requires that $2\alpha - 3\delta - 2r + d - 1 = 2 - 2\delta < 0$; thus

$$\int_{\mathbb{R}^d} \omega_{s,r,\delta}(\xi)|F(\xi)|^2\,\mathrm{d}\xi \asymp \sum_{k=K}^\infty 4^{2\alpha k}4^{-k(3\delta-2r+d-1)} = \sum_{k=K}^\infty 4^{k(2-2\delta)} < \infty. \tag{2.31}$$

Because $r \leq 1$ we have that $2\alpha - \delta - d + 1 = 2 - 2r \geq 0$; thus

$$\|F\|^2_{L^2} = \int_{\mathbb{R}^d} |F(\xi)|^2\,\mathrm{d}\xi \asymp \sum_{k=K}^\infty 4^{2\alpha k}4^{-k(\delta+d-1)} = \sum_{k=K}^\infty 4^{2-2r} = \infty. \tag{2.32}$$

Finally, α is defined so that $2\alpha - 1 + 2r - d - \delta = 0$, and so

$$\|F\circ Q\|^2_{X^s_{r,\delta}} = \int_{\mathbb{R}^d} \omega_{s,r,\delta}(\xi)|F(Q\xi)|^2\,\mathrm{d}\xi \asymp \sum_{k=K}^\infty 4^{2\alpha k}4^{-k(1-2r+\delta+d)} = \sum_{k=K}^\infty 1 = \infty. \tag{2.33}$$

With this, set $f = \check{F}$. The previous calculations imply that $f \in X^s_{r,\delta}(\mathbb{R}^d)$ but $f \notin L^2(\mathbb{R}^d) \subseteq H^s(\mathbb{R}^d)$ and

$f \circ Q \notin X^s_{r,\delta}(\mathbb{R}^d)$. F is compactly supported and in L^1, so we conclude that $f \in C_0^\infty(\mathbb{R}^d)$. ■

We close out this section with identifying when derivatives of $X^s_{r,\delta}$ functions lie in classical Sobolev spaces. We give some examples of the uses of these estimates in the last section.

Proposition 2.9. Suppose that $1-r \le s$, $\delta - r \le \tau$, and $\sigma \le s$ and $f \in X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$. Suppose $\varphi : \mathbb{R}^d \to [0,\infty)$ satisfies $\varphi(\xi) \asymp |\xi|^\tau$ for $|\xi| \le 1$ and $\varphi(\xi) \asymp |\xi|^\sigma$ for $|\xi| \ge 1$. We then have $\varphi\left(\sqrt{-\Delta}\right) f \in H^{s-\tau}(\mathbb{R}^d;\mathbb{F})$ and $\partial_1 f \in \dot{H}^{-r}(\mathbb{R}^d;\mathbb{F})$ and

$$\left\|\varphi\left(\sqrt{-\Delta}\right) f\right\|_{H^{s-\tau}} + \|\partial_1 f\|_{\dot{H}^{-r}} \lesssim \|f\|_{X^s_{r,\delta}}. \tag{2.34}$$

In particular, when $\delta - r \le \tau \le s$, we have that $\left\|(-\Delta)^{\tau/2} f\right\|_{H^{s-\tau}} + \|\partial_1 f\|_{\dot{H}^{-r}} \lesssim \|f\|_{X^s_{r,\delta}}$.

Proof. We note since $\delta - r \le \tau$ that $|\xi|^{2\tau} \le |\xi|^{2(\delta - r)} \le \omega_{s,r,\delta}(\xi)$ for $|\xi| \le 1$. Thus we have $(1+|\xi|^2)^{s-\sigma}|\xi|^{2\tau} \lesssim \omega_{s,r,\delta}(\xi)$ for $|\xi| \le 1$. Clearly we have $(1+|\xi|^2)^{s-\sigma}|\xi|^{2\sigma} \lesssim |\xi|^{2s} \asymp \omega_{r,s,\delta}(\xi)$ for $|\xi| \ge 1$, thus we see

$$\begin{aligned}\left\|\varphi\left(\sqrt{-\Delta}\right) f\right\|^2_{H^{s-t}} &\asymp \int_{B(0,1)} (1+|\xi|^2)^{s-\sigma}|\xi|^{2\tau}\left|\hat{f}(\xi)\right|^2 \,\mathrm{d}\xi + \int_{B(0,1)^c} (1+|\xi|^2)^{s-\sigma}|\xi|^{2\sigma}\left|\hat{f}(\xi)\right|^2 \,\mathrm{d}\xi \\ &\lesssim \int \omega_{s,r,\delta}(\xi)\left|\hat{f}(\xi)\right|^2 \,\mathrm{d}\xi = \|f\|^2_{X^s_{r,\delta}}.\end{aligned} \tag{2.35}$$

The inclusion and estimate for $\partial_1 f$ follow similarly after we observe that $1-r \le s$ implies that $\frac{|\xi_1|^2}{|\xi|^{2r}} \le |\xi|^{2s}$ for $|\xi| \ge 1$. The final claim follows by setting $\varphi(\xi) = |\xi|^\tau$. ■

3 When is $X^s_{r,\delta}$ an algebra?

We now proceed to our goal of characterizing when $X^s_{r,\delta}$ is an algebra. Our approach is modeled on the Littlewood-Paley techniques used in [3]. We will look to leverage that H^s is an algebra when $s > d/2$. The first step shows that for $f \in X^s_{r,\delta}$ and $g \in H^s$, we know that $fg \in H^s$.

Theorem 3.1. Assume that $d > 1 + \delta - 2r$ and $s > d/2$. Then the following hold.

(a) There exists a constant $C > 0$ such that if $f \in X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ and $g \in H^s(\mathbb{R}^d;\mathbb{F})$, then $fg \in H^s(\mathbb{R}^d;\mathbb{F})$ and

$$\|fg\|_{H^s} \le C\|f\|_{X^s_{r,\delta}}\|g\|_{H^s}. \tag{3.1}$$

(b) For $1 \le k \in \mathbb{N}$ the map

$$H^s(\mathbb{R}^d;\mathbb{F}) \times \prod_{j=1}^k X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}) \ni (g, f_1, \ldots, f_k) \mapsto g\prod_{j=1}^k f_j \in H^s(\mathbb{R}^d;\mathbb{F}) \tag{3.2}$$

is a bounded $(k+1)$–linear map.

Proof. The second item follows easily from the first, so we only prove the first. Fix $f \in X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ and write $f = f_l + f_h = f_{l,1} + f_{h,1}$ as in Theorem 2.3 with $R = 1$. Since $f_h \in H^s(\mathbb{R}^d;\mathbb{F})$ and $s > d/2$, we have that $f_h g \in H^s(\mathbb{R}^d;\mathbb{F})$ and $\|f_h g\|_{H^s} \lesssim \|f_h\|_{H^s}\|g\|_{H^s} \lesssim \|f\|_{X^s_{r,\delta}}\|g\|_{H^s}$.

On the other hand, since $f_l \in C^k_b(\mathbb{R}^d;\mathbb{F})$ for every $k \in \mathbb{N}$, we may readily deduce that multiplication by f_l defines a bounded linear map from $H^k(\mathbb{R}^d;\mathbb{F})$ to itself for every $k \in \mathbb{N}$, and $\|f_l\varphi\|_{H^k} \lesssim \|f_l\|_{C^k_b}\|\varphi\|_{H^k} \lesssim \|f\|_{X^s_{r,\delta}}\|\varphi\|_{H^k}$ for every $\varphi \in H^k(\mathbb{R}^d;\mathbb{F})$. Interpolating, we conclude that multiplication by f_l defines a bounded linear map from $H^t(\mathbb{R}^d;\mathbb{F})$ to itself for every $0 \le t \in \mathbb{R}$ and $\|f_l\varphi\|_{H^t} \le C(t)\|f\|_{X^s_{r,\delta}}\|\varphi\|_{H^k}$ for every $\varphi \in H^t(\mathbb{R}^d;\mathbb{F})$, where $C(t) \ge 0$ is a constant that depends on t but is independent of f or φ. Picking $t = s$ then shows that $f_l g \in H^s(\mathbb{R}^d;\mathbb{F})$ and $\|f_l g\|_{H^s} \lesssim \|f\|_{X^s_{r,\delta}}\|g\|_{H^s}$. ■

We now mark some notation for convenience.

Definition 3.2. We define the measurable function $\mu_{r,\delta} : \mathbb{R}^d \to [0,\infty)$ by $\mu_{r,\delta}(\xi) = \frac{|\xi_1|+|\xi|^\delta}{|\xi|^r}$, which is asymptotically equivalent to $\sqrt{\omega_{s,r,\delta}(\xi)}$ for $|\xi| \le 1$. We then have that

$$\|f\|_{X^s_{r,\delta}} \asymp \left\|\left(\mu_{r,\delta}\chi_{B(0,1)} + \langle\cdot\rangle^s\chi_{B(0,1)^c}\right)\hat{f}\right\|_{L^2}.$$

We then define the trilinear functional $I : \left(L^0(\mathbb{R}^d;[0,\infty])\right)^3 \to [0,\infty]$ by

$$I(F,G,H) = \int_{B(0,1)^2} \frac{\mu_{r,\delta}(\xi+\eta)}{\mu_{r,\delta}(\xi)\mu_{r,\delta}(\eta)} F(\xi)G(\eta)H(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{3.3}$$

where $L^0(\mathbb{R}^d;[0,\infty])$ denotes the nonnegative measurable functions on $\mathbb{R}^d$.

In fact, I induces a bounded trilinear functional over $\left(L^2(\mathbb{R}^d;\mathbb{F})\right)^3$ as long as I is bounded over $\left(L^2(\mathbb{R}^d;[0,\infty])\right)^3$.

Lemma 3.3. Suppose there exists a constant $C > 0$ such that

$$I(F,G,H) \le C\|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2} \tag{3.4}$$

for all $F,G,H \in L^2(\mathbb{R}^d;[0,\infty])$. Then I induces a bounded trilinear map over $\left(L^2(\mathbb{R}^d;\mathbb{F})\right)^3$ into $\mathbb{F}$ satisfying the same formula, and there exists some constant $C' > 0$ such that

$$|I(F,G,H)| \le C'\|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2} \tag{3.5}$$

Proof. The proof for this is identical to Lemma 2.7 in [5] . ■

Using this lemma, we can identify a crucial link between $X^s_{r,\delta}$ being an algebra and the boundedness of I.

Proposition 3.4. Assume that $d > 1 + \delta - 2r$, $r \le 1$, and $s > d/2$. There exists a constant $C > 0$ such that

$$\|fg\|_{X^s_{r,\delta}} \le C\|f\|_{X^s_{r,\delta}}\|g\|_{X^s_{r,\delta}} \text{ for all } f,g \in X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F}) \tag{3.6}$$

if and only if there exists a constant $C > 0$ such that

$$I(F,G,H) = \int_{B(0,1)^2} \frac{\mu_{r,\delta}(\xi+\eta)}{\mu_{r,\delta}(\xi)\mu_{r,\delta}(\eta)} F(\xi)G(\eta)H(\xi+\eta)\,\mathrm{d}\xi\mathrm{d}\eta \le C\|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2} \tag{3.7}$$

for all $F,G,H \in L^2(\mathbb{R}^d;[0,\infty])$.

Proof. First, suppose that I is bounded and let $f,g \in X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$. We then set $f_0 = \left(\chi_{B(0,1)}\hat{f}\right)^{\vee} \in C^\infty_0(\mathbb{R}^d;\mathbb{F})$ and $f_1 = \left(\chi_{B(0,1)^c}\hat{f}\right)^{\vee} = f - f_0 \in H^s(\mathbb{R}^d;\mathbb{F})$. We define g_0 and g_1 similarly for g. We then have

$$fg = f_0g_0 + f_0g_1 + f_1g_0 + f_1g_1. \tag{3.8}$$

By Theorem 2.7, Proposition 3.1, and the fact that $H^s(\mathbb{R}^d;\mathbb{F})$ is an algebra, whenever $i+j \ge 1$ we have that f_ig_j is supported in $B(0,1)^c$, is in $H^s \hookrightarrow X^s_{r,\delta}$ and

$$\left\|f_ig_j\right\|_{X^s_{r,\delta}} \asymp \left\|f_ig_j\right\|_{H^s} \lesssim \|f_i\|_{X^s_{r,\delta}}\left\|g_j\right\|_{X^s_{r,\delta}} \le \|f\|_{X^s_{r,\delta}}\|g\|_{X^s_{r,\delta}}. \tag{3.9}$$

Thus, it only remains to analyze f_0g_0. Theorem 2.3 shows that $\hat{f}_0$ and $\hat{g}_0$ are integrable and supported in $B(0,1)$, so Young's inequality implies $\hat{f}_0 * \hat{g}_0 \in L^1(\mathbb{R}^d;\mathbb{F})$ and $\operatorname{supp}(\hat{f}_0 * \hat{g}_0) \subseteq B(0,2)$. Let $\varphi \in \mathscr{S}(\mathbb{R}^d;\mathbb{F})$. We employ Tonelli's theorem to calculate

$$\begin{aligned}\int_{\mathbb{R}^d} \mu_{r,\delta}\left(\hat{f}_0 * \hat{g}_0\right)\varphi &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mu_{r,\delta}(\xi+\eta)\hat{f}_0(\xi)\hat{g}_0(\eta)\varphi(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \int_{B(0,1)^2} \mu_{r,\delta}(\xi+\eta)\hat{f}_0(\xi)\hat{g}_0(\eta)\varphi(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = I(\mu_{r,\delta}\hat{f}_0, \mu_{r,\delta}\hat{f}_0, \varphi).\end{aligned} \tag{3.10}$$

By the assumed boundedness of I we have

$$\left|\int_{\mathbb{R}^d} \mu_{r,\delta}(\hat{f}_0 * \hat{g}_0)\varphi\right| \lesssim \left\|\mu_{r,\delta}\hat{f}_0\right\|_{L^2}\left\|\mu_{r,\delta}\hat{g}_0\right\|_{L^2}\|\varphi\|_{L^2} \lesssim \|f\|_{X^s_{r,\delta}}\|g\|_{X^s_{r,\delta}}\|\varphi\|_{L^2}. \tag{3.11}$$

By the density of $\mathscr{S}$ in L^2, we see that the left hand side extends to define a bounded linear functional on L^2 obeying the same estimate, and so the Riesz representation theorem tells us that $\mu_{r,\delta}(\hat{f}_0 * \hat{g}_0) \in L^2(\mathbb{R}^d)$ and

$$\|f\|_{X^s_{r,\delta}}\|g\|_{X^s_{r,\delta}} \gtrsim \left\|\mu_{r,\delta}(\hat{f}_0 * \hat{g}_0)\right\|_{L^2} = \left\|\mu_{r,\delta}\widehat{f_0g_0}\right\|_{L^2} \asymp \|f_0g_0\|_{X^s_{r,\delta}} \tag{3.12}$$

The last bound followed because $\hat{f}_0 * \hat{g}_0$ is compactly supported. We thus have the desired result.

Conversely, assume that $X^s_{r,\delta}(\mathbb{R}^d)$ is an algebra. Let $F, G, H \in L^2(\mathbb{R}^d; [0,\infty])$ and note that $I(F,G,H) = I(F\chi_{B(0,1)}, G\chi_{B(0,1)}, H)$ due to the domain of the integral, and the fact that we have $\left(\chi_{B(0,1)}F/\mu_{r,\delta}\right)^\vee$ and $\left(\chi_{B(0,1)}G/\mu_{r,\delta}\right)^\vee$ are both in $X^s_{r,\delta}$. Thus by Cauchy-Schwarz and the boundedness of products in $X^s_{r,\delta}$ we have

$$
\begin{aligned}
I(F,G,H) &= \int_{\mathbb{R}^d} \mu_{r,\delta}\left(\left(\chi_{B(0,1)}F/\mu_{r,\delta}\right) * \left(\chi_{B(0,1)}G/\mu_{r,\delta}\right)\right)H \\
&\leq \left\|\mu_{r,\delta}\left(\left(\chi_{B(0,1)}F/\mu_{r,\delta}\right) * \left(\chi_{B(0,1)}G/\mu_{r,\delta}\right)\right)\right\|_{L^2}\|H\|_{L^2} = \left\|\left(\chi_{B(0,1)}F/\mu_{r,\delta}\right)^\vee\left(\chi_{B(0,1)}G/\mu_{r,\delta}\right)^\vee\right\|_{X^s_{r,\delta}}\|H\|_{L^2} \\
&\lesssim \left\|\left(\chi_{B(0,1)}F/\mu_{r,\delta}\right)^\vee\right\|_{X^s_{r,\delta}}\left\|\left(\chi_{B(0,1)}G/\mu_{r,\delta}\right)^\vee\right\|_{X^s_{r,\delta}}\|H\|_{L^2} \lesssim \|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2} \qquad (3.13)
\end{aligned}
$$

By Lemma 3.3, we then have the desired result. ∎

Thus to prove that $X^s_{r,\delta}$ is a Banach algebra, we wish to analyze the boundedness of the functional I. To do this, we first split the domain $B(0,1)^2$ into two sets to get control of I in each independently.

Definition 3.5. We partition $B(0,1)^2$ into two sets E_0 and E_1 as follows:

$$
E_0 = \{(\xi,\eta) \in B(0,1)^2 |\xi| + |\eta| \leq 3||\xi| - |\eta||\} \text{ and } E_1 = \{(\xi,\eta) \in B(0,1)^2 |\xi| + |\eta| > 3||\xi| - |\eta||\} \qquad (3.14)
$$

From this we write I as a sum of two operators I_0 and I_1, where for $i \in \{0,1\}$ we have

$$
I_i = \int_{E_i} \frac{\mu_{r,\delta}(\xi+\eta)}{\mu_{r,\delta}(\xi)\mu_{r,\delta}(\eta)} F(\xi)G(\eta)H(\xi+\eta)\, \mathrm{d}\xi \mathrm{d}\eta \qquad (3.15)
$$

We now analyze the boundedness of I_0 and I_1 in turn.

Proposition 3.6. $(\xi,\eta) \in E_1 \Leftrightarrow (\xi,\eta) \in B(0,1)^2$ and $\frac{1}{2}|\eta| < |\xi| < 2|\eta|$. Additionally, $(\xi,\eta) \in E_1$ implies $|\xi+\eta| < 3|\eta|$.

Proof. The proof for this can be found in Lemma 2.10 of [5]. ∎

Lemma 3.7. If $(\xi,\eta) \in E_0$, then $\mu_{r,\delta}(\xi+\eta) \lesssim \mu_{r,\delta}(\xi) + \mu_{r,\delta}(\eta)$

Proof. We note since $\delta - r \geq 0$, we have that $(|\xi| + |\eta|)^{\delta-r} \lesssim |\xi|^{\delta-r} + |\eta|^{\delta-r}$. Using this, the triangle inequality, and the estimate from the definition of E_0, we have

$$\mu_{r,\delta}(\xi+\eta) = \frac{\left|\xi_1+\eta_1\right|}{\left|\xi+\eta\right|^r} + \left|\xi+\eta\right|^{\delta-r} \leq \frac{|\xi_1|+\left|\eta_1\right|}{\left||\xi|-|\eta|\right|^r} + (|\xi|+|\eta|)^{\delta-r}$$

$$\lesssim \frac{|\xi_1|+\left|\eta_1\right|}{(|\xi|+|\eta|)^r} + |\xi|^{\delta-r} + |\eta|^{\delta-r} \leq \mu_{r,\delta}(\xi)+\mu_{r,\delta}(\eta) \quad (3.16)$$

■

Proposition 3.8. $I_0(F,G,H) \lesssim \|1/\mu_{r,\delta}\|_{L^2}\|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2}$

Proof. The proof is identical to that of Proposition 2.12 in [5]. ■

3.1 Splitting E_1 further

We now aim to get more control of I_1. To accomplish this, we localize further in E_1.

Definition 3.9. Suppose we have $m,n \in \mathbb{N}$. Then we define

$$E_{m,n} = \{(\xi,\eta) \in E_1\, 2^{-m-1} < |\xi| \leq 2^{-m} \text{ and } 2^{-n+1} < |\xi+\eta| \leq 2^{-n+2}\}. \quad (3.17)$$

Similar to before, we also define

$$I_{m,n} = \int_{E_{m,n}} \frac{\mu_{r,\delta}(\xi+\eta)}{\mu_{r,\delta}(\xi)\mu_{r,\delta}(\eta)} F(\xi)G(\eta)H(\xi+\eta)\,\mathrm{d}\xi\mathrm{d}\eta \quad (3.18)$$

In addition, we define the annulus $A = B[0,4] \setminus B(0,1/2)$ and set

$$F_m = F\chi_{2^{-m-1}A}, G_m = G\chi_{2^{-m-1}A}, H_n = H\chi_{2^{-n}A}. \quad (3.19)$$

for $F,G,H \in L^2(\mathbb{R}^d,[0,\infty])$.

Lemma 3.10. We have

$$\bigcup_{m=0}^{\infty}\bigcup_{n=m}^{\infty} E_{m,n} = E_1 \text{ and } \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} I_{m,n} = I_1 \quad (3.20)$$

Proof. The proof is identical to that of Lemma 2.14 in [5]. ■

We now come to our first major bound. For large d, we will see that in fact we can get control of I_1, thus proving that $X^s_{r,\delta}$ is an algebra. For small dimension, we will need to do some further splitting.

Proposition 3.11. Let $r \leq 1$. Let $F,G,H \in L^2(\mathbb{R}^d,[0,\infty])$. The following hold.

(a) If $d \geq 4\delta - 2r - 2$, then

$$I_1(F,G,H) \lesssim \|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2} \tag{3.21}$$

(b) If $d < 4\delta - 2r - 2$, then

$$\sum_{m=0}^{\infty}\sum_{n>km}^{\infty} I_{m,n}(F,G,H) \lesssim \|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2} \tag{3.22}$$

for $k = \frac{2(\delta-r)}{1-r+d/2}$.

Proof. Let $m,n \in \mathbb{N}$ and $n \geq m$. Then we have

$$I_{m,n}(F,G,H) \leq I_1(F_m,G_m,H_n) = \int_{E_1} \frac{\mu_{r,\delta}(\xi+\eta)}{\mu_{r,\delta}(\xi)\mu_{r,\delta}(\eta)} F_m(\xi)G_m(\eta)H_n(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{3.23}$$

The right hand integral vanishes except when $2^{-m-2} \leq |\xi|, |\eta| \leq 2^{-m+1}$, and $2^{-n-1} \leq |\xi+\eta| \leq 2^{-n+2}$. Thus we have the estimates

$$\mu_{r,\delta}(\xi) = \frac{|\xi_1| + |\xi|^\delta}{|\xi|^r} \geq |\xi|^{\delta-r} \gtrsim 2^{-m(\delta-r)}, \tag{3.24}$$

and similarly $\mu_{r,\delta}(\eta) \gtrsim 2^{-m(\delta-r)}$. Furthermore, we get

$$\mu_{r,\delta}(\xi+\eta) = \frac{|\xi_1+\eta_1| + |\xi+\eta|^\delta}{|\xi+\eta|^r} \leq \frac{|\xi+\eta| + |\xi+\eta|^\delta}{|\xi+\eta|^r} \lesssim \frac{2^{-n} + 2^{-n\delta}}{2^{-nr}} \lesssim 2^{-n(1-r)}. \tag{3.25}$$

Thus combining the estimates, we get

$$\frac{\mu_{r,\delta}(\xi+\eta)}{\mu_{r,\delta}(\xi)\mu_{r,\delta}(\eta)} \lesssim \frac{2^{-n(1-r)}}{2^{-2m(\delta-r)}} = 2^{2m(\delta-r)-n(1-r)}, \tag{3.26}$$

and so we find

$$I_{m,n}(F,G,H) \lesssim 2^{2m(\delta-r)-n(1-r)} \int_{E_1} F_m(\xi)G_m(\eta)H_n(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{3.27}$$

For $\ell \in \mathbb{Z}^d$, let Q_ℓ be the closed cube centered at $2^{-n}\ell$ of side length 2^{-n} and $\tilde{Q}_\ell$ denote the closed cube centered at $-2^{-n}\ell$ of side length $9 \cdot 2^{-n}$. We note then that

$$\max_{\ell \in \mathbb{Z}^d} \left\|\chi_{Q_\ell}\right\|_{L^2} = \left\|\chi_{Q_0}\right\|_{L^2} = 2^{-nd/2}. \tag{3.28}$$

Note if $(\xi,\eta) \in E_1$ and $\xi \in Q_\ell$, then

$$\left|\eta + 2^{-n}\ell\right|_\infty \leq \left|\eta+\xi\right| + |-\xi + 2^{-n}\ell|_\infty \leq 2^{-n+2} + 2^{-n-1} = (9/2)2^{-n}, \tag{3.29}$$

and so $\eta \in \tilde{Q}_\ell$. Thus we compute

$$\int_{E_1} F_m(\xi)G_m(\eta)H_n(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \le \sum_{\ell\in\mathbb{Z}^d}\int_{B(0,1)^2}\left(F_m\chi_{Q_\ell}\right)(\xi)\left(G_m\chi_{\tilde{Q}_\ell}\right)(\eta)H_n(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$\le \sum_{\ell\in\mathbb{Z}^d}\int_{B(0,1)}\left(F_m\chi_{Q_\ell}\right)(\xi)\left\|G_m\chi_{\tilde{Q}_\ell}\right\|_{L^2}\|H_n\|_{L^2}\,\mathrm{d}\xi \le \|H_n\|_{L^2}\left\|\chi_{Q_0}\right\|_{L^2}\sum_{\ell\in\mathbb{Z}^d}\left\|F_n\chi_{Q_\ell}\right\|_{L^2}\left\|G_m\chi_{\tilde{Q}_\ell}\right\|_{L^2}$$

$$\le 2^{-nd/2}\|H_n\|_{L^2}\left(\int_{\mathbb{R}^d}|F_m(\xi)|^2\sum_{\ell\in\mathbb{Z}^d}\chi_{Q_l}(\xi)\,\mathrm{d}\xi\right)^{1/2}\left(\int_{\mathbb{R}^d}\left|G_m(\eta)\right|^2\sum_{\ell\in\mathbb{Z}^d}\chi_{\tilde{Q}_l}(\eta)\,\mathrm{d}\eta\right)$$

$$\lesssim 2^{-nd/2}\|F_m\|_{L^2}\|G_m\|_{L^2}\|H_n\|_{L^2}. \quad (3.30)$$

Synthesizing these, we obtain

$$I_{m,n}(F,G,H) \lesssim 2^{2m(\delta-r)-n(1-r+d/2)}. \quad (3.31)$$

Now we break to cases based on dimension. If $d \ge 4\delta - 2r - 2$, then we simply bound

$$\sum_{m\ge0}\sum_{n\ge m} I_{m,n}(F,G,H) \lesssim \sum_{m\ge0}\sum_{n\ge m}2^{2m(\delta-r)-n(1-r+d/2)}\|F_m\|_{L^2}\|G_m\|_{L^2}\|H_n\|_{L^2}$$

$$\le \sum_{m=0}^{\infty}2^{2m(\delta-r)}\|F_m\|_{L^2}\|G_m\|_{L^2}\left(\sum_{n=m}^{\infty}2^{-n(2-2r+d)}\right)^{1/2}\left(\int|H|^2\sum_{n=m}^{\infty}\chi_{2^{-n}A}\right)^{1/2}$$

$$\lesssim \|H\|_{L^2}\sum_{m=0}^{\infty}2^{m(2\delta-r-1-d/2)}\|F_m\|_{L^2}\|G_m\|_{L^2} \lesssim \|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2}. \quad (3.32)$$

On the other hand, if $d < 4\delta - 2r - 2$, then we split further: for $k = \frac{2(\delta-r)}{1-r+d/2}$ an analogous argument shows that

$$\sum_{m\ge0}\sum_{n\ge km} I_{m,n}(F,G,H) \lesssim \|H\|_{L^2}\sum_{m=0}^{\infty}2^{m(2\delta-2r-k+kr-kd/2)}\|F_m\|_{L^2}\|G_m\|_{L^2} \lesssim \|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2}. \quad (3.33)$$

■

We now prove our final bound when d is small.

Proposition 3.12. Suppose that $r \le 1$ and $d < 4\delta - 2r - 2$, and if $d = 2$ further suppose that $\delta \le 2$. For $F, G, H \in L^2(\mathbb{R}^d, [0,\infty])$ we have the estimate

$$\sum_{m\ge0}\sum_{m\le n\le km} I_{m,n}(F,G,H) \lesssim \|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2}, \quad (3.34)$$

where $k = \frac{2(\delta-r)}{1-r+d/2}$.

Proof. We define

$$R_{p,\pi}(\alpha) = 2^{-\delta m}[-\alpha/2 + p, \alpha/2 + p] \times 2^{-n} \prod_{k=2}^{d} [-\alpha/2 + \pi_{k-1}, \alpha/2 + \pi_{k-1}] \tag{3.35}$$

For $\xi \in R_{p,\pi}(1)$ and $\eta \in R_{q,\sigma}(1)$, we get

$$2^{-n}(\pi_{k-1} + \sigma_{k-1} - 1/2) \le \xi_k + \eta_k \le 2^{-n}(\pi_{k-1} + \sigma_{k-1} - 1/2) \tag{3.36}$$

Combining with the fact that $|\xi + \eta| \le 2^{-n+2}$, we get

$$|\pi_{k-1} + \sigma_{k-1}| \le |\pi_{k-1} + \sigma_{k-1} - 2^n(\xi_k + \eta_k)| + |2^n(\xi_k + \eta_k)| \le 9/2 \tag{3.37}$$

In particular, $\eta \in R_{q,-\pi}(9)$. Combined with $\xi \in R_{p,\pi}(1)$, we get $\xi + \eta \in R_{p+q,0}(10)$. Thus

$$\begin{aligned} &I_{m,n}(F,G,H) \\ &\quad \le \sum_{p,q\in\mathbb{Z}} \sum_{\pi,\sigma\in\mathbb{Z}^{d-1}} \int_{E_1} \frac{\mu_{r,\delta}(\xi+\eta)}{\mu_{r,\delta}(\xi)\mu_{r,\delta}(\eta)} (F_m \chi_{R_{p,\pi}}(1))(\xi)(G_m \chi_{R_{q,-\pi}}(9))(\eta)(H_n \chi_{R_{p+q,0}}(10))(\xi+\eta)\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{3.38}$$

Now if $\xi \in R_{p,\pi}(1) \cap 2^{-m-1}A$, then $2^{\delta m}|\xi_1| \ge |p| - 1/2$, so

$$\mu_{r,\delta}(\xi) = \frac{|\xi_1| + |\xi|^\delta}{|\xi|^r} \gtrsim \frac{2^{-\delta m}|p| + 2^{-\delta m}}{2^{-mr}} \gtrsim 2^{-m(\delta - r)}|p| \tag{3.39}$$

and similarly

$$\mu_{r,\delta}(\eta) \gtrsim 2^{-m(\delta-r)}|q|. \tag{3.40}$$

Then because $\xi + \eta \in R_{p+q,0}(10) \cap 2^{-n-1}A$, we get

$$\begin{aligned} \mu(\xi+\eta) = \frac{|\xi_1+\eta_1| + |\xi+\eta|^\delta}{|\xi+\eta|^r} &\lesssim 2^{nr}(|\xi_1+\eta_1| + 2^{-n\delta}) \\ &= 2^{nr-m\delta}(2^{m\delta}|\xi_1+\eta_1| + 2^{\delta(m-n)}) \lesssim 2^{nr-m\delta}(|p|+|q|) \end{aligned} \tag{3.41}$$

Putting everything together, we get

$$\frac{\mu_{r,\delta}(\xi+\eta)}{\mu_{r,\delta}(\xi)\mu_{r,\delta}(\eta)} \lesssim 2^{nr+m(\delta-2r)}\left(\frac{1}{|p|} + \frac{1}{|q|}\right) \tag{3.42}$$

Then for fixed $m, n \in \mathbb{N}$, we see

$$\begin{aligned}
&2^{-nr-m(\delta-2r)} I_{m,n}(F,G,H) \\
&\quad \leq \sum_{p,q\in\mathbb{Z}} \sum_{\pi\in\mathbb{Z}^{d-1}} \int_{B(0,1)} \left(\frac{1}{|p|}+\frac{1}{|q|}\right)(F_m\chi_{R_{p,\pi}(1)})(\xi) \int_{B(0,1)} (G_m\chi_{R_{q,-\pi}(9)})(\eta)(H_n\chi_{R_{p+q,0}(10)})(\xi+\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi \\
&\qquad \leq \sum_{p,q\in\mathbb{Z}} \sum_{\pi\in\mathbb{Z}^{d-1}} \int_{B(0,1)} \left(\frac{1}{|p|}+\frac{1}{|q|}\right) F_m(\xi)\chi_{R_{p,\pi}(1)}(\xi) \left\|G_m\chi_{R_{q,-\pi}(9)}\right\|_{L^2} \left\|H_n\chi_{R_{p+q,0}(10)}\right\|_{L^2} \mathrm{d}\xi \\
&\qquad \leq \sum_{p,q\in\mathbb{Z}} \sum_{\pi\in\mathbb{Z}^{d-1}} \left(\frac{1}{|p|}+\frac{1}{|q|}\right) \left\|F_m\chi_{R_{p,\pi}(1)}\right\|_{L^2} \left\|\chi_{R_{p,\pi}(1)}\right\|_{L^2} \left\|G_m\chi_{R_{q,-\pi}(9)}\right\|_{L^2} \left\|H_n\chi_{R_{p+q,0}(10)}\right\|_{L^2} \\
&\qquad = 2^{(-\delta m-(d-1)n)/2} \sum_{p,q\in\mathbb{Z}} \sum_{\pi\in\mathbb{Z}^{d-1}} \left(\frac{1}{|p|}+\frac{1}{|q|}\right) \left\|F_m\chi_{R_{p,\pi}(1)}\right\|_{L^2} \left\|G_m\chi_{R_{q,-\pi}(9)}\right\|_{L^2} \left\|H_n\chi_{R_{p+q,0}(10)}\right\|_{L^2} \qquad (3.43)
\end{aligned}$$

Firstly, we handle the sum over $\pi \in \mathbb{Z}^{d-1}$. Indeed, we find for each $p, q \in \mathbb{Z}$

$$\begin{aligned}
\sum_{\pi\in\mathbb{Z}^{d-1}} \left\|F_m\chi_{R_{p,\pi}(1)}\right\|_{L^2} \left\|G_m\chi_{R_{q,-\pi}(9)}\right\|_{L^2} &\leq \left(\int_{\mathbb{R}^d} |F_m|^2 \sum_{\pi\in\mathbb{Z}^{d-1}} \chi_{R_{p,\pi}(1)}\right)^{1/2} \left(\int_{\mathbb{R}^d} |G_m|^2 \sum_{\pi\in\mathbb{Z}^{d-1}} \chi_{R_{q,-\pi}(9)}\right)^{1/2} && (3.44) \\
&\leq \left\|F_m\chi_{\bigcup_{\pi\in\mathbb{Z}^{d-1}} R_{p,\pi}(1)}\right\|_{L^2} \left\|G_m\chi_{\bigcup_{\pi\in\mathbb{Z}^{d-1}} R_{q,-\pi}(9)}\right\|_{L^2} && (3.45)
\end{aligned}$$

Now we consider the sums over p, q. First we consider the term containing $\frac{1}{\max\{1,|p|\}}$. We find

$$\sum_{p,q\in\mathbb{Z}} \frac{1}{|p|} \left\|F_m\chi_{\bigcup_{\pi\in\mathbb{Z}^{d-1}} R_{p,\pi}(1)}\right\|_{L^2} \left\|G_m\chi_{\bigcup_{\pi\in\mathbb{Z}^{d-1}} R_{q,-\pi}(9)}\right\|_{L^2} \left\|H_n\chi_{R_{p+q,0}(10)}\right\|_{L^2} \qquad (3.46)$$

$$\lesssim \left\|G_m\chi_{\bigcup_{q\in\mathbb{Z}}\bigcup_{\pi\in\mathbb{Z}^{d-1}} R_{q,-\pi}(9)}\right\|_{L^2} \sum_{p\in\mathbb{Z}} \frac{1}{\max\{1,|p|\}} \left\|F_m\chi_{\bigcup_{\pi\in\mathbb{Z}^{d-1}} R_{p,\pi}(1)}\right\|_{L^2} \left\|H_n\chi_{\bigcup_{q\in\mathbb{Z}} R_{p+q,0}(10)}\right\|_{L^2} \qquad (3.47)$$

$$\leq \|G_m\|_{L^2}\|H_n\|_{L^2} \left(\sum_{p\in\mathbb{Z}} \frac{1}{\max\{1,|p|^2\}}\right)^{1/2} \left\|F_m\chi_{\bigcup_{p\in\mathbb{Z}}\bigcup_{\pi\in\mathbb{Z}^{d-1}} R_{p,\pi}(1)}\right\|_{L^2} \lesssim \|F_m\|_{L^2}\|G_m\|_{L^2}\|H_n\|_{L^2} \qquad (3.48)$$

Analogously, we can compute the sum containing $\frac{1}{\max\{1,|q|\}}$ to find

$$\sum_{p,q\in\mathbb{Z}} \frac{1}{|q|} \left\|F_m\chi_{\bigcup_{\pi\in\mathbb{Z}^{d-1}} R_{p,\pi}(1)}\right\|_{L^2} \left\|G_m\chi_{\bigcup_{\pi\in\mathbb{Z}^{d-1}} R_{q,-\pi}(9)}\right\|_{L^2} \left\|H_n\chi_{R_{p+q,0}(10)}\right\|_{L^2} \lesssim \|F_m\|_{L^2}\|G_m\|_{L^2}\|H_n\|_{L^2} \qquad (3.49)$$

Combining the previous bounds, we then arrive at the estimate

$$I_{m,n}(F,G,H) \lesssim 2^{n\left(r-\frac{d}{2}+\frac{1}{2}\right)+m\left(\frac{\delta}{2}-2r\right)} \|F_m\|_{L^2}\|G_m\|_{L^2}\|H_n\|_{L^2}. \qquad (3.50)$$

We now split to two cases. In the first assume that $r - \frac{d}{2} + \frac{1}{2} \leq 0$, looking at the inner sum in (3.34) we bound using the first term to find

$$\sum_{n=m}^{km} I_{m,n}(F,G,H) \lesssim 2^{m\left(\frac{\delta}{2}-2r\right)}\left(\sum_{n=m}^{km} 2^{n(2r-d+1)}\right)^{1/2} \|H\|_{L^2}\|F_m\|_{L^2}\|G_m\|_{L^2}$$
$$\lesssim 2^{m\left(\left(\frac{\delta}{2}-2r\right)+\left(r-\frac{d}{2}+\frac{1}{2}\right)\right)}\|H\|_{L^2}\|F_m\|_{L^2}\|G_m\|_{L^2} \quad (3.51)$$

Now summing over m, this converges if and only if $\frac{\delta-d+1}{2} \le r$ which is true by hypothesis.

For the other case, we have $r - \frac{d}{2} + \frac{1}{2} > 0$. In particular, since we have already restricted $r \le 1$ this implies that $d = 2$. Plugging in $d = 2$ in (3.50), we again analyze the inner sum of (3.34), this time bounding using the last term to see

$$\sum_{n=m}^{km} I_{m,n}(F,G,H) \lesssim 2^{m\left(\frac{\delta}{2}-2r\right)}\left(\sum_{n=m}^{km} 2^{n(2r-2+1)}\right)^{1/2} \|H\|_{L^2}\|F_m\|_{L^2}\|G_m\|_{L^2}$$
$$\lesssim 2^{m\left(\left(\frac{\delta}{2}-2r\right)+k(r-1/2)\right)}\|H\|_{L^2}\|F_m\|_{L^2}\|G_m\|_{L^2} \lesssim 2^{m\left(\left(\frac{\delta}{2}-2r\right)+\frac{2(\delta-r)}{2-r}(r-1/2)\right)}\|H\|_{L^2}\|F_m\|_{L^2}\|G_m\|_{L^2} \quad (3.52)$$

Again summing over m, this converges when

$$\left(\frac{\delta}{2}-2r\right)+\frac{2(\delta-r)}{2-r}(r-1/2) \le 0 \Leftrightarrow (2-r)\left(\frac{\delta}{2}-2r\right)+(\delta-r)(2r-1) \le 0$$
$$\Leftrightarrow \frac{\delta}{2} - 2r - \frac{r\delta}{2} + 2r^2 + \frac{d\delta}{4} - rd + 2\delta r - 2r^2 - 2\delta + 2r + \delta - r \le 0 \Leftrightarrow \delta \le 2, \quad (3.53)$$

the latter condition of which is assumed to hold by hypothesis. ∎

In fact, the assumption that $\delta \le 2$ when $d = 2$ was necessary, as the next proposition shows.

Proposition 3.13. Suppose that $d = 2$, $r > 0$, and $\delta > 2$. Then there does not exist a constant $C > 0$ such that

$$\int_{B(0,1)^2} \frac{\mu_{r,\delta}(\xi+\eta)}{\mu_{r,\delta}(\xi)\mu_{r,\delta}(\eta)} F(\xi)G(\eta)H(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \le C\|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2} \quad (3.54)$$

for every $F, G, H \in L^2(\mathbb{R}^2)$.

Proof. Define the sets $Q = B_\infty((2^{-m\delta}, 2^{-m}), 2^{-m\delta-2})$, $Q' = B_\infty((2^{-m\delta}, -2^{-m}), 2^{-m\delta-2})$, and $P = Q+Q' = B_\infty((2^{-m\delta}, 0), 2^{-m\delta-1}$

We then note for $\xi \in Q$, we have that $|\xi| \asymp 2^{-m}$, and so

$$\mu_{r,\delta}(\xi) = \frac{|\xi_1| + |\xi|^\delta}{|\xi|^r} \lesssim \frac{2^{-m\delta} + 2^{-m\delta}}{2^{-mr}} \lesssim 2^{-m(\delta-r)}. \quad (3.55)$$

and similarly $\mu_{r,\delta}(\eta) \lesssim 2^{-m(\delta-r)}$ when $\eta \in Q'$. We also note $\xi + \eta \in P$ when $\xi \in Q, \eta \in Q'$, and so $|\xi| \asymp |\eta| \asymp$

$2^{-m\delta}$ and $|\xi+\eta| \asymp 2^{-m\delta}$. Thus we see

$$\mu_{r,\delta}(\xi+\eta) = \frac{|\xi_1+\eta_1| + |\xi+\eta|^\delta}{|\xi+\eta|^r} \gtrsim \frac{2^{-m\delta} + 2^{-m\delta^2}}{2^{-m\delta r}} \gtrsim 2^{-m\delta(1-r)}. \tag{3.56}$$

Finally, we note that $\mu(Q) \asymp \mu(Q') \asymp \mu(P) \asymp 2^{-2m\delta}$ and so $\|\chi_Q\|_{L^2}\|\chi_{Q'}\|_{L^2}\|\chi_P\|_{L^2} \asymp 2^{-3m\delta}$. On the other hand, we can compute

$$\begin{aligned}\iint_{B(0,1)^2} \frac{\mu_{r,\delta}(\xi+\eta)}{\mu_{r,\delta}(\xi)\mu_{r,\delta}(\eta)} \chi_Q(\xi)\chi_{Q'}(\eta)\chi_P(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta &= \iint_{Q\times Q'} \frac{\mu_{r,\delta}(\xi+\eta)}{\mu_{r,\delta}(\xi)\mu_{r,\delta}(\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\gtrsim 2^{-m(\delta(1-r)-2(\delta-r))}\mu(Q)\mu(Q') \gtrsim 2^{-m(4\delta+\delta(1-r)-2(\delta-r))}. \end{aligned} \tag{3.57}$$

In particular, if the linear functional was bounded, we could find some $C>0$ such that

$$2^{-m(4\delta+\delta(1-r)-2(\delta-r)-3\delta)} \le C \tag{3.58}$$

for all $m \in \mathbb{N}$. However, since $\delta > 2$ and $r > 0$ we know that

$$4\delta + \delta(1-r) - 2(\delta - r) - 3\delta = r(2-\delta) < 0, \tag{3.59}$$

and so making m arbitrarily large, we contradict the inequality. ■

We can now state our main result.

Theorem 3.14. Suppose $d > 1 + \delta - 2r$, $r \le 1$, $s > d/2$. If $d \ge 3$, then $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is an algebra. If $d = 2$, then $X^s_{r,\delta}$ is an algebra if and only if $\delta \le 2$.

Proof. The $d \ge 3$ and $d = 2, \delta \le 2$ cases follows by combining Propositions 3.4, 3.8, 3.11, 3.12. For $d = 2$, $\delta > 2$, we note that we must have $r > 1/2 > 0$ due to our other restrictions, and so Proposition 3.13 shows that $X^s_{r,\delta}$ is not an algebra in this case. ■

4 PDE applications

In this section we give a couple short and simple applications of the $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ spaces in constructing traveling wave solutions to PDEs. Recall the motivating pseudodifferential equation equation from the introduction: $\mathrm{d}tv + (-\Delta)^{\delta/2} = F$, which reduces to $-\gamma\partial_1 u + (-\Delta)^{\delta/2}u = f$ after the traveling wave reformulation. We assume here that $\gamma \ne 0$, which corresponds to actual traveling wave solutions and not stationary solutions. Here the operator $(-\Delta)^{\delta/2}$ comes from a homogeneous function on the Fourier side, namely $\mathbb{R}^d \ni \xi \mapsto (2\pi|\xi|)^\delta \in [0,\infty)$.

In fact, the spaces $X^s_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ are designed to be more flexible by handling more general symbols with a manifest "bihomogeneity," meaning possibly different homogeneous behavior for small and large frequencies. To describe this, we let $\varphi : \mathbb{R}^d \to [0,\infty)$ be a continuous function such that

$$\varphi(\xi) \asymp \begin{cases} C_0|\xi|^\delta & \text{for } |\xi| \ll 1 \\ C_1|\xi|^\sigma & \text{for } |\xi| \gg 1 \end{cases} \tag{4.1}$$

for $\delta,\sigma \in \mathbb{R}$ satisfying $\delta > 1$ and $\sigma \in \mathbb{R}$. We write $D = \sqrt{-\Delta}$ and $\varphi(D)$ for the pseudodifferential operator acting via $\widehat{\varphi(D)u}(\xi) = \varphi(\xi)\hat{u}(\xi)$. As two particular examples: (1) the function $\varphi(\xi) = (2\pi|\xi|)^\delta$ gives $\varphi(D) = (-\Delta)^{\delta/2}$ from the introduction, and satisfies $\delta = \sigma$; (2) the function $\varphi(\xi) = |\xi|\tanh(|\xi|)$ is of this type with $\delta = 2$ and $\sigma = 1$; this particular φ arose in the analysis in [8] and is related to the classic gravity-wave dispersion relation. We can then consider the modification of the previous pseudodifferential equation: $\partial_t v + \varphi(D)u = F$, which reduces to $-\gamma\partial_1 u + \varphi(D)u = f$ after the traveling wave reformulation. Our first result establishes solvability of this linear problem.

Theorem 4.1. Let $s,r,\delta,\sigma \in \mathbb{R}$ satisfy $1 < \delta$, $d > 1+\delta-2r$, $r \geq 0$, $\sigma \leq s$, and $1-r \leq s$. Suppose $\varphi : \mathbb{R}^d \to [0,\infty)$ is a continuous function satisfying (4.1). Let $\beta,\gamma \in \mathbb{R}\backslash\{0\}$. Then the map $-\gamma\partial_1 + \beta\varphi(D) : X^{s+\sigma}_{r,\delta}(\mathbb{R}^d;\mathbb{F}) \to (H^s \cap \dot{H}^{-r})(\mathbb{R}^d;\mathbb{F})$ is well-defined and induces a bounded linear isomorphism. In particular, for each $f \in (H^s \cap \dot{H}^{-r})(\mathbb{R}^d;\mathbb{F})$ there exists a unique $u \in X^{s+\sigma}_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ solving

$$-\gamma\partial_1 u + \beta\varphi(D)u = f. \tag{4.2}$$

Proof. Since $r \geq 0$, we have $\delta - r \leq \delta$, and by hypothesis we have $\sigma \leq s$, so Proposition 2.9 shows that $-\gamma\partial_1$ and $\varphi(D)$ are both bounded linear operators from $X^{s+\sigma}_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ to $(H^s \cap \dot{H}^{-r})(\mathbb{R}^d;\mathbb{F})$. Consider, then, the problem of finding u satisfying (4.2) for a given f. Applying the Fourier transform, we see that this is equivalent to

$$[-\gamma 2\pi i\xi_1 + \beta\varphi(\xi)]\hat{u}(\xi) = \hat{f}(\xi) \text{ for a.e. } \xi \in \mathbb{R}^d. \tag{4.3}$$

If a solutions u exists with $f = 0$, then since the term in brackets on the left only vanishes at most on a null set, we must have that $\hat{u} = 0$ a.e., and hence $u = 0$. Thus, the linear map is injective. We also learn from this that it is surjective, as we may use this equation to define $\hat{u}$ in terms of $\hat{f}$, and then

$$\|u\|^2_{X^{s+\sigma}_{r,\delta}} = \int_{B(0,1)} \frac{|\xi_1|^2 + |\xi|^{2\delta}}{|\xi|^{2r}} |\hat{u}(\xi)|^2 d\xi + \int_{B(0,1)^c} |\xi|^{2(s+\sigma)} |\hat{u}(\xi)|^2 d\xi$$
$$\asymp \int_{B(0,1)} \frac{1}{|\xi|^{2r}} |\hat{f}(\xi)|^2 d\xi + \int_{B(0,1)^c} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi = \|f\|^2_{H^s \cap \dot{H}^{-r}}, \quad (4.4)$$

which shows that u indeed belongs to $X^{s+\sigma}_{r,\delta}(\mathbb{R}^d;\mathbb{F})$. Hence, the linear map is an isomorphism. ■

Next we give an extremely simple but instructive example of how the isomorphism from the previous theorem can be used to solve nonlinear variants of the above traveling wave problem. Note that the u we obtain from this theorem gives a traveling wave solution by setting $v(x,t) = u(x - \gamma e_1 t)$.

Theorem 4.2. Suppose $\varphi : \mathbb{R}^d \to [0,\infty)$ is a continuous function satisfying (4.1). Let $s, r, \delta, \sigma \in \mathbb{R}$ satisfy $1 < \delta$, $d > 1 + \delta - 2r$, $r \geq 0$, $\sigma \leq s$, $1 - r \leq s$, and $s + \sigma > d/2$. If $d = 2$, further suppose that $\delta \leq 2$. Suppose that $R > 0$ is such that the ball $B(0,R) \subseteq \mathbb{F}$ is the ball of convergence for two analytic functions $\zeta, \psi : B(0,R) \to \mathbb{F}$ such that $\zeta(0) = \psi(0) = 0$ and $\zeta'(0) = \alpha \in \mathbb{R}\backslash\{0\}$ and $\psi'(0) = \beta \in \mathbb{R}\backslash\{0\}$. Then there exists an open set $\varnothing \neq \mathcal{U} \subseteq (H^s \cap \dot{H}^{-r})(\mathbb{R}^d;\mathbb{F})$ such that for each $f \in \mathcal{U}$ there exists a unique $u \in X^{s+\sigma}_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ satisfying

$$-\gamma \partial_1 [\zeta(u)] + \varphi(D)\psi(u) = f. \quad (4.5)$$

Moreover, the induced map $\mathcal{U} \ni f \mapsto u \in X^{s+\delta}_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is analytic.

Proof. We begin by noting that since $s + \sigma > d/2$, Proposition 2.3 shows that $X^{s+\sigma}_{r,\delta}(\mathbb{R}^d;\mathbb{F}) \hookrightarrow C^0_b(\mathbb{R}^d;\mathbb{F})$. Theorem 3.14 shows that $X^{s+\sigma}_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ is an algebra, but it does not show that it is a Banach algebra. However, by rescaling the norm on $X^{s+\sigma}_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ by a fixed constant we may assume without loss of generality that $\|uv\|_{X^{s+\sigma}_{r,\delta}} \leq \|u\|_{X^{s+\sigma}_{r,\delta}} \|v\|_{X^{s+\sigma}_{r,\delta}}$. We may then select an open set $0 \in \mathcal{V} \subseteq X^{s+\sigma}_{r,\delta}(\mathbb{R}^d;\mathbb{F})$ such that if $u \in \mathcal{V}$ then $u(\mathbb{R}^d) \subseteq B(0,R)$. Thus, $\zeta \circ u$ and $\psi \circ u$ are well-defined for $u \in \mathcal{V}$, and this induces analytic maps $\zeta, \psi : \mathcal{V} \to X^{s+\sigma}_{r,\delta}(\mathbb{R}^d;\mathbb{F})$.

Proposition 2.9 and the above show that the map $N : \mathcal{V} \to (H^s \cap \dot{H}^{-r})(\mathbb{R}^d;\mathbb{F})$ defined by $N(u) = -\gamma\partial_1\zeta(u) + \varphi(D)\psi(u)$ is well-defined and analytic, and by construction $N(0) = 0$ and its derivatives satisfies $\mathbf{D}N(0)v = -\alpha\gamma\partial_1 v + \beta\varphi(D)v$. This linear map is an isomorphism thanks to Theorem 4.1, and so we may apply the inverse function theorem (see, for instance, Theorem 10.2.5 in [2]) to conclude. ■

By a similar argument, we can also prove the following variant, which is a nonlinear "divergence form" version of the problem from the introduction.

Theorem 4.3. Let $0 \leq s, r \in \mathbb{R}$ and $1 < \delta \in \mathbb{R}$ satisfy $d > 1 + \delta - 2r$ and $s > d/2$. If $d = 2$, further suppose that $\delta \leq 2$. Suppose that $R > 0$ is such that the ball $B(0, R) \subseteq \mathbb{F}$ is the ball of convergence for two analytic functions $\zeta, \psi : B(0, R) \to \mathbb{F}$ such that $\zeta(0) = \psi(0) = 0$, $\zeta'(0) = \alpha \in \mathbb{R} \setminus \{0\}$, and $\psi'(0) = \beta \in \mathbb{R} \setminus \{0\}$. Then there exists an open set $\varnothing \neq \mathcal{U} \subseteq (H^s \cap \dot{H}^{-r})(\mathbb{R}^d; \mathbb{F})$ such that for each $f \in \mathcal{U}$ there exists a unique $u \in X^{s+\delta}_{r,\delta}(\mathbb{R}^d; \mathbb{F})$ satisfying

$$-\gamma \partial_1[\zeta(u)] - (-\Delta)^{\delta/2-1} \operatorname{div}[(1 + \psi(u))\nabla u] = f. \tag{4.6}$$

Moreover, the induced map $\mathcal{U} \ni f \mapsto u \in X^{s+\delta}_{r,\delta}(\mathbb{R}^d; \mathbb{F})$ is analytic.

Part II

Extending the Cauchy integral

5 Uniform Function Spaces

5.1 Definitions and Preliminaries

From now on, we work in the uniformity of uniform convergence. We first define some useful subsets of $\mathcal{F}(X;Z)$ that we will be employing going forward.

Definitions 5.1. Let X be a set and $(Z,\mathscr{U})$ be a uniform space. We define the following spaces:

(a) $\mathcal{TB}(X;Z) = \{f \in \mathcal{F}(X;Z) \mid f[X] \text{ is totally bounded}\}$, the totally bounded functions from X to Z.

(b) $\mathcal{S}(X;Z) = \{f \in \mathcal{F}(X;Z) \mid f[X] \text{ is finite}\}$, the simple functions from X to Z.

To write simple functions in an explicit way, we set some notation.

Definition 5.2. Let X and Z be sets. Suppose $A \subseteq X$ and $z \in Z$. We define $\chi_A^z = A \times \{z\}$. In other words, $\chi_A^z : X \rightharpoonup Z$ is the partial function from X to Z with $\mathrm{dom}(\chi_A^z) = A$ and $\chi_A^z(x) = z$ for all $x \in A$.

With this notation in hand, we note that for every simple function $s \in \mathcal{S}(X;Z)$, we can find a partition $A_1, \ldots, A_m$ of X and points $z_1, \ldots, z_m \in Z$ such that $s = \bigcup_{k \leq m} \chi_{A_k}^{z_k}$. This union defines a total function precisely when $A_1, \ldots, A_m$ partitions X.

5.2 Properties of the uniformity of uniform convergence

Proposition 5.3. Let X be a set.

(a) Suppose (Z,d) is an extended pseudometric space with associated uniformity $\mathscr{U}$. We define

$$d_u(f,g) = \sup_{x \in X} d(f(x), g(x)).$$

Then d_u is an extended pseudometric on $\mathcal{F}(X;Z)$ compatible with $\widetilde{\mathscr{U}}$. Furthermore, if Z is a true metric space, then d_u restricted to $\mathcal{B}(X;Z)$ is also a true metric.

(b) Suppose Z is a topological group. Then $\mathcal{F}(X;Z)$ is also a topological group equipped with the group operation $fg(x) = f(x)g(x)$.

(c) Suppose Z is a topological vector space. Then $\mathcal{F}(X;Z)$ is also a topological vector space. Furthermore, if Z is locally convex defined by seminorms $\{[\cdot]_\alpha\}_\alpha$, then $\mathcal{F}(X;Z)$ is also locally convex with seminorms defined by $\{f \mapsto \sup_{x\in X}[f(x)]_\alpha\}_\alpha$ and if Z is normed with norm $\|\cdot\|$, then $\mathcal{F}(X;Z)$ is also normed with norm $f \mapsto \sup_{x\in X}\|f(x)\|$.

Proof. All of these are routine verifications that follow readily from the definitions, so we leave them as an exercise. ∎

Proposition 5.4. Let X be a set and $(Z,\mathscr{U})$ be a uniform space.

(a) If Z is complete, then $\mathcal{F}(X;Z)$ is also complete.

(b) $\mathcal{TB}(X;Z)$ is a closed subset of $\mathcal{F}(X;Z)$. In particular, when Z is complete, $\mathcal{TB}(X;Z)$ is also complete.

Proof.

(a) Suppose Z is complete. Then let $(f_\alpha)_\alpha$ be a Cauchy net in $\mathcal{F}(X;Z)$. Since uniform convergence implies pointwise convergence everywhere, we then know for each $x \in X$ that $(f_\alpha(x))_\alpha$ is a Cauchy net in Z and thus convergent to some z_x. We then define $f : X \to Z$ by

$$f(x) = z_x$$

for each $x \in X$. We now claim that $(f_\alpha)_\alpha$ converges uniformly to f. Towards that end, let $U \in \mathscr{U}$ and pick $V \in \mathscr{U}$ such that $V^2 \subseteq U$. Pick β such that for all $\alpha, \alpha' \geq \beta$ we have that $(f_\alpha(x), f_{\alpha'}(x)) \in V$ for all $x \in X$. Now let $x \in X$ and $\alpha \geq \beta$ be arbitrary. Pick $\alpha' \geq \beta$ such that $(f(x), f_{\alpha'}(x)) \in V$. We then see that $(f(x), f_\alpha(x)) \in V^2 \subseteq U$. Our choice of α, x and U was arbitrary, so we obtain that $(f_\alpha)_\alpha$ converges uniformly to f as desired.

(b) Let $(f_\alpha)_\alpha$ be a net in $\mathcal{TB}(X;Z)$ converging to some $f \in \mathcal{F}(X;Z)$ uniformly. Let $U \in \mathscr{U}$ and pick $V \in \mathscr{U}$ such that $V^2 \subseteq U$. Then pick some α large enough such that $(f_\alpha(x), f(x)) \in V$ for all $x \in X$. Pick some $z_1, \ldots, z_m \in Z$ and $n \in \mathbb{N}$ such that $f_\alpha[X] \subseteq \bigcup_{k\leq m} V[z_k]$. But then for each $x \in X$ we have $(f_\alpha(x), z_k) \in V$ for some k ,s o $(f(x), z_k) \in V^2 \subseteq U$. In particular, we find that $f[X] \subseteq \bigcup_{k\leq m} U[y_k]$ and so $f[X]$ is also totally bounded as desired. ∎

In fact, we know that $\mathcal{S}(X;Z)$ and $\mathcal{TB}(X;Z)$ are related - the latter is the uniform closure of the former.

Proposition 5.5. $\overline{\mathcal{S}(X;Z)} = \mathcal{TB}(X;Z)$.

Proof. Firstly, let $f \in \overline{\mathcal{S}(X;Z)}$ and let $U \in \mathscr{U}$ be arbitrary. Pick $s \in \mathcal{S}(X;Z)$ such that $(s(x), f(x)) \in U$ for all $x \in X$. Then $f[X] \subseteq U[s[X]]$ and so f is totally bounded as desired.

On the other hand, suppose we have $f \in \mathcal{TB}(X;Z)$. Pick $z_1, \ldots, z_n$ such that $f[X] \subseteq \bigcup U[z_k]$. Let $A_1 = f^{-1}[U[z_1]]$ and recursively define $A_k = f^{-1}[U[z_k]] \setminus A_{k-1}$. Then $A_1, \ldots, A_n$ partitions X, so we define $s : X \to Z$ as

$$s = \bigcup_{k=1}^{m} \chi_{A_k}^{z_k}.$$

Then by construction we see that for each $x \in X$ we can pick some k such that $x \in A_k$ and we find $f(x) \in U[z_k]$, so $(f(x), s(x)) = (f(x), z_k) \in U$ which tells us s is U-close to f as desired. ■

Proposition 5.6. Let X and Y be sets and $(Z, \mathscr{U})$ be a uniform space. Define the natural currying map $T : \mathcal{F}(X;\mathcal{F}(Y;Z)) \to \mathcal{F}(X \times Y; Z)$ as $Tf(x,y) = f(x)(y)$ for each $f \in \mathcal{F}(X;\mathcal{F}(Y;Z))$. Then T is a uniform homeomorphism.

Proof. It's obvious that T is a bijection, so it suffices to prove that both T and T^{-1} are uniformly continuous. However, seeing this is easy - let U be an entourage in Z, $\widetilde{U}$ be the corresponding entourage in $\mathcal{F}(Y;Z)$ (i.e. $(f,g) \in \widetilde{U}$ if and only if $(f(x), g(x)) \in U$ for all $x \in X$) and $\widetilde{\widetilde{U}}$ be the corresponding entourage in $\mathcal{F}(X;\mathcal{F}(Y;Z))$. In addition, let $\widehat{U}$ be the corresponding entourage in $\mathcal{F}(X \times Y; Z)$. Then we see that

$$\begin{aligned}
(f,g) \in (T \times T)\left[\widetilde{\widetilde{U}}\right] &\Leftrightarrow (T^{-1}f, T^{-1}g) \in \widetilde{\widetilde{U}} && (5.1)\\
&\Leftrightarrow (T^{-1}f(x), T^{-1}g(x)) \in \widetilde{U} \ \forall x \in X && (5.2)\\
&\Leftrightarrow (T^{-1}f(x)(y), T^{-1}g(x)(y)) \in U \ \forall x \in X \ \forall y \in Y && (5.3)\\
&\Leftrightarrow (f(x,y), g(x,y)) \in U \ \forall (x,y) \in X \times Y \Leftrightarrow (f,g) \in \widehat{U} && (5.4)
\end{aligned}$$

and so $(T \times T)\left[\widetilde{\widetilde{U}}\right]$ is also an entourage in $\mathcal{F}(X \times Y; Z)$. An analogous argument shows that $(T^{-1} \times T^{-1})\left[\widehat{U}\right]$ is an entourage in $\mathcal{F}(X;\mathcal{F}(Y;Z))$, and so we see that T is a uniform homeomorphism as desired. ■

6 Regulated Functions

Let's start adding some more structure in working towards the integral. In what follows, X will be a set and $\mathscr{A}$ will be an algebra on X. $(Z, \mathscr{U})$ will be a uniform space as before.

Definition 6.1. Let $(X, \mathscr{A})$ be a set equipped with an algebra. We say that $A_1, \ldots, A_n$ is an *$\mathscr{A}$-partition* of

X if each $A_k \in \mathscr{A}$, $A_k \cap A_j = \varnothing$ for $j \neq k$ and $\bigcup_{k=1}^{n} A_k = X$.

Definition 6.2. We say that $f \in \mathcal{S}(X;Z)$ is $\mathscr{A}$-simple if for each $z \in Z$ we know that $f^{-1}[\{z\}] \in \mathscr{A}$. We then define $\mathcal{S}_{\mathscr{A}}(X;Z) = \{f \in \mathcal{S}(X;Z) \mid f \text{ is } \mathscr{A}\text{-simple}\}$. With this, we set $\mathcal{R}_{\mathscr{A}}(X;Z) = \overline{\mathcal{S}_{\mathscr{A}}(X;Z)}$ where the closure is taken in the uniformity defined on $\mathcal{F}(X;Y)$ in the previous section. When $f \in \mathcal{R}_{\mathscr{A}}(X;Y)$, we say that f is *$\mathscr{A}$-regulated*.

Indeed, this is the natural generalization of Bourbaki's definition of regulated.

Example 6.3. Let $a, b \in \mathbb{R}$ with $a < b$ and set $\mathscr{A} = \left\{\bigcup_{k=1}^{n} I_k \;\middle|\; I_k \subseteq [a,b] \text{ is convex}\right\}$. It is easy to see that $\mathscr{A}$ is an algebra. With this, for any uniform space Z, we see that $\mathcal{R}_{\mathscr{A}}([a,b];Z)$ defined as above is precisely the set that Bourbaki called the regulated functions.

We see that the notion of regulated behaves naturally when we take subsets of our codomain or subalgebras.

Proposition 6.4. Let $(X, \mathscr{A})$ be a set-algebra pair and let $(Z, \mathscr{U})$ be a uniform space. Suppose that $\mathscr{B} \subseteq \mathscr{A}$ is a subalgebra of $\mathscr{A}$ and that $Y \subseteq Z$. Then $\mathcal{R}_{\mathscr{B}}(X;Y) \subseteq \mathcal{R}_{\mathscr{A}}(X;Z)$ where Y is endowed with the subspace uniformity.

Proof. Clearly if f is $\mathscr{B}$-simple, then f must also be $\mathscr{A}$-simple since $\mathscr{B} \subseteq \mathscr{A}$. Additionally, it is obvious that $\mathcal{S}_{\mathscr{B}}(X;Y) \subseteq \mathcal{S}_{\mathscr{B}}(X;Z)$. Since closures preserve inclusion we have

$$\mathcal{R}_{\mathscr{B}}(X;Y) = \overline{\mathcal{S}_{\mathscr{B}}(X;Y)} \subseteq \overline{\mathcal{S}_{\mathscr{B}}(X;Z)} \subseteq \overline{\mathcal{S}_{\mathscr{A}}(X;Z)} = \mathcal{R}_{\mathscr{A}}(X;Z). \qquad \blacksquare$$

This then allows us to readily conclude that all $\mathscr{A}$-regulated functions are totally bounded.

Corollary 6.5. Let $(X, \mathscr{A})$ be a set-algebra pair and let $(Z, \mathscr{U})$ be a uniform space. Then $\mathcal{R}_{\mathscr{A}}(X;Z) \subseteq \mathcal{TB}(X;Z)$.

Proof. By Proposition 5.5, we know that $\mathcal{R}_{\mathcal{P}(X)}(X;Z) = \overline{\mathcal{S}_{\mathcal{P}(X)}(X;Z)} = \overline{\mathcal{S}(X;Z)} = \mathcal{TB}(X;Z)$. Then $\mathscr{A}$ is a subalgebra of $\mathcal{P}(X)$, so the previous proposition gives the desired result. $\blacksquare$

We now present an alternate, more concrete characterization of being $\mathscr{A}$-regulated.

Lemma 6.6. Let $f \in \mathcal{F}(X;Z)$. The following are equivalent:

(a) $f \in \mathcal{R}_{\mathscr{A}}(X;Z)$

(b) For every $U \in \mathscr{U}$, there exists an $\mathscr{A}$-partition $A_1, \ldots, A_n \in \mathscr{A}$ such that for each k we have that $(f(x_1), f(x_2)) \in U$ for all $x_1, x_2 \in A_k$.

Proof. First suppose f is $\mathscr{A}$-regulated. Then pick a net of simple functions $(s_\alpha)_\alpha$ in $\mathcal{S}_{\mathscr{A}}(X;Z)$ such that $s_\alpha \to f$ uniformly. Let $U \in \mathscr{U}$ be arbitrary and pick $V \in \mathscr{U}$ with $V^2 \subseteq U$. Pick β large enough such that for all $\alpha \geq \beta$, we have $(s_\alpha(x), f(x)) \in V$ for all $x \in X$. Write $s_\beta = \sum_{i=1}^n y_i \chi_{A_i}$. Then for each i and $x_1, x_2 \in A_i$, we have that $(s_\beta(x_2), f(x_1)) = (s_\beta(x_1), f(x_1)) \in V$ and $(s_\beta(x_2), f(x_2)) \in V$, so we have $(f(x_1), f(x_2)) \in V^2 \subseteq U$, as desired.

On the other hand, suppose (b) holds. Let $U \in \mathscr{U}$. Pick $A_1, \ldots, A_n$ as in (b) corresponding to U. Then for each k, pick some $x_k \in A_k$ and define $s \in \mathcal{S}_{\mathscr{A}}(X;Z)$ by

$$s(x) = f(x_k)$$

for each $x \in A_k$. With this, let $x \in X$ be arbitrary and pick A_k such that $x \in A_k$. Then $(f(x), s(x)) = (f(x), f(x_k)) \in U$. Our choice of U was arbitrary, so we have $f \in \mathcal{R}_{\mathscr{A}}(X;Z)$ as desired. ■

6.1 Properties of Regulated Functions

It will often be useful to restrict the codomains of the simple functions we use to approximate a regulate function. In fact, we can always take the simple functions to take values in $f[X]$. We prove this now.

Proposition 6.7. Let $(X, \mathscr{A})$ be a set equipped with an algebra and $(Z, \mathscr{U})$ be a uniform space. Given $f \in \mathcal{R}_{\mathscr{A}}(X;Z)$, there exists a net of simple functions $(s_\alpha)_\alpha$ such that $s_\alpha \to f$ uniformly and for each $\alpha \in A$ we have that $s_\alpha[X] \subseteq f[X]$.

Proof. Pick any net of simple functions $(t_\alpha)_\alpha$ such that $t_\alpha \to f$ uniformly. Fix α and enumerate $t_\alpha[X]$ as $z_1^{(\alpha)}, \ldots, z_{n_\alpha}^{(\alpha)}$. We will define a new function s_α as follows: for each $1 \leq k \leq n_\alpha$, if $z_k^{(\alpha)} \in f[X]$, then we do nothing and set $s_\alpha(x) = z_k^{(\alpha)}$ for all $x \in t_\alpha^{-1}\left\{z_k^{(\alpha)}\right\}$. Otherwise, we choose some $\tilde{z}_k^{(\alpha)} \in f\left[t_\alpha^{-1}\left\{z_k^{(\alpha)}\right\}\right]$ and define $s_\alpha(x) = \tilde{z}_k^{(\alpha)}$ for all $x \in t_\alpha^{-1}\left\{z_k^{(\alpha)}\right\}$. In this way, it's clear that $s_\alpha[X] \subseteq f[X]$.

It remains to show that $s_\alpha \to f$ uniformly. Towards that end, let U be an entourage in Y. Pick an entourage V such that $V^2 \subseteq U$ and let β be such that for all $\alpha \geq \beta$ we have that $(t_\alpha(x), f(x)) \in V$ for all $x \in X$. With this, let $\alpha \geq \beta$ and $x \in X$ be arbitrary. Pick $1 \leq k \leq n_\alpha$ such that $x \in t_\alpha^{-1}\left\{z_k^{(\alpha)}\right\}$ and pick some $y \in f^{-1}\left\{\tilde{z}_k^{(\alpha)}\right\} \cap t_\alpha^{-1}\left\{z_k^{(\alpha)}\right\}$. Then $(s_\alpha(x), t_\alpha(x)) = \left(\tilde{z}_k^{(\alpha)}, z_k^{(\alpha)}\right) = (f(y), t_\alpha(y)) \in V$ and so $(s_\alpha(x), f(x)) \in V^2 \subseteq U$, which gives the desired result. ■

Similarly, we can also restrict the domain of our simple functions.

Proposition 6.8. Let $(X, \mathscr{A})$ be a set-algebra pair and $(Z, \mathscr{U})$ be a uniform space and suppose $f \in \mathcal{R}_{\mathscr{A}}(X; Z)$ with $(s_\alpha)_\alpha$ approaching f uniformly. Suppose $z \in Z$ and $\{x \in X \mid f(x) \neq z\} \subseteq A$ for some $A \in \mathscr{A}$. Then $\left(s_\alpha \restriction_A \cup \chi^z_{A^c}\right)_\alpha$ approaches f uniformly.

Proof. Let $U \subseteq Z$ be an entourage and pick β large enough such that for all $\alpha \geq \beta$ we have $(s_\alpha(x), f(x)) \in U$ for all $x \in X$. Then for $x \in A$, we have $(s_\alpha \restriction_A(x), f(x)) \in U$, and for $x \notin A$ we have $(\chi^z_A(x), f(x)) = (z, z) \in U$. Thus we have the desired result. ■

Corollary 6.9. Let $(X, \mathscr{A})$ be a set-algebra pair and $(Z, \mathscr{U})$ be a uniform space and suppose $f \in \mathcal{R}_{\mathscr{A}}(X; Z)$ with $(s_\alpha)_\alpha$ approaching f uniformly. Suppose $\{x \in X \mid f(x) \neq 0\} \subseteq A$ for some $A \in \mathscr{A}$. Then $(s_\alpha \chi_A)_\alpha$ approaches f uniformly.

We now turn our attention to compositions. Given $(X, \mathscr{A})$ and $(Y, \mathscr{B})$, and a $\mathscr{B}$-regulated function $f : Y \to Z$ we can ask for which functions φ the inner composition $f \circ \varphi$ is $\mathscr{A}$-regulated. For this, we first make a definition.

Definition 6.10. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be spaces equipped with algebras. We say that $\varphi : X \to Y$ is $\mathscr{A}$-$\mathscr{B}$ measurable if $\varphi^{-1}[B] \in \mathscr{A}$ for each $B \in \mathscr{B}$.

Proposition 6.11. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be spaces equipped with algebras and $(Z, \mathscr{U})$ be a uniform space. Suppose $\varphi : X \to Y$ is $\mathscr{A}$-$\mathscr{B}$ measurable. If $f \in \mathcal{R}_{\mathscr{B}}(Y; Z)$, then $f \circ \varphi \in \mathcal{R}_{\mathscr{A}}(X; Z)$.

Proof. Let U be an entourage in Z. Since f is $\mathscr{B}$-regulated, we can find $B_1, \ldots, B_n \in \mathscr{B}$ that cover Y and such that $f[B_k] \times f[B_k] \subseteq U$ for each $1 \leq k \leq n$. Then $\varphi^{-1}[B_k] \in \mathscr{A}$ for all k, and $\varphi^{-1}[B_1], \ldots, \varphi^{-1}[B_n]$ covers X since the B_k-s cover Y. Furthermore, we see

$$(f \circ \varphi)\left[\varphi^{-1}[B_k]\right] \times (f \circ \varphi)\left[\varphi^{-1}[B_k]\right] \subseteq f[B_k] \times f[B_k] \subseteq U$$

and so $f \circ \varphi$ is also $\mathscr{A}$-regulated as desired. ■

Thus inner compositions are regulated when our function φ interacts nicely with the set algebra structure. We can also guess that an outer composition $\varphi \circ f$ is regulated when our function φ interacts nicely with the uniform structure, and indeed this is the case.

Lemma 6.12. Let X be a set and $(Y,\mathscr{U}_Y)$ and $(Z,\mathscr{U}_Z)$ be uniform spaces. Suppose $\varphi : Y \to Z$ is uniformly continuous. Then if $(f_\alpha)_\alpha$ in $\mathcal{F}(X;Y)$ converges uniformly to some $f \in \mathcal{F}(X;Y)$, we have that $(\varphi \circ f_\alpha)_\alpha$ converges uniformly to $\varphi \circ f \in \mathcal{F}(X;Z)$.

Proof. Suppose $(f_\alpha)_\alpha$ in $\mathcal{F}(X;Y)$ converges uniformly to f as above and let $U \in \mathscr{U}_Z$ be arbitrary. Pick $V \in \mathscr{U}_Y$ such that whenever $(y_1,y_2) \in V$ we know that $(\varphi(y_1),\varphi(y_2)) \in U$. Then pick β such that for all $\alpha \geq \beta$ we have that $(f_\alpha(x), f(x)) \in V$ for all $x \in X$. Then $(\varphi \circ f_\alpha(x), \varphi \circ f(x)) \in U$ for all such x and α, so in fact $\varphi \circ f_\alpha$ converges uniformly to $\varphi \circ f$ as desired. ∎

Corollary 6.13. Let $(X,\mathscr{A})$ be a set-algebra pair and $(Y,\mathscr{U}_Y)$ and $(Z,\mathscr{U}_Z)$ be uniform spaces. Suppose $\varphi : Y \to Z$ is uniformly continuous and $f : X \to Y$ is $\mathscr{A}$-regulated. Then $\varphi \circ f$ is $\mathscr{A}$-regulated.

We now note that if D is dense in the codomain, then this density carries over to density of simple functions taking values in D.

Proposition 6.14. Let $(X,\mathscr{A})$ be a set equipped with algebra and let $(Z,\mathscr{U})$ be a uniform space. Suppose $D \subseteq Z$ is dense in Z. Then $\mathcal{S}_{\mathscr{A}}(X;D)$ is dense in $\mathcal{R}_{\mathscr{A}}(X;Z)$.

Proof. It suffices to show that $\mathcal{S}_{\mathscr{A}}(X;D)$ is dense in $\mathcal{S}_{\mathscr{A}}(X;Z)$ since the latter is dense in $\mathcal{R}_{\mathscr{A}}(X;Z)$. Towards that end, let $U \in \mathscr{U}$ and $f \in \mathcal{S}_{\mathscr{A}}(X;Z)$, with

$$f = \sum_{k=1}^{n} z_k \chi_{A_k}$$

with $z_k \in Z$ and $A_1,\ldots,A_n$ forming an $\mathscr{A}$-partition of X. Then for each z_k, pick some $d_k \in D$ such that $(z_k, d_k) \in U$. Then define

$$g = \sum_{k=1}^{n} d_k \chi_{A_k}.$$

We then see that $g \in \mathcal{S}_{\mathscr{A}}(X;D)$. By construction, we have $(f(x), g(x)) \in U$ for all $x \in X$, i.e. $(f,g) \in \widetilde{U}$. Our choice of U was arbitrary so we have the desired result. ∎

When our codomain is a topological vector space, we know that the regulated functions form a vector space and so are closed under addition and scalar multiplication. However, we can also ask about bilinear mappings and products of regulated functions. In fact, such mappings remain regulated, as we prove now.

Proposition 6.15. Let Y, Z, and W be topological vector spaces and $(X, \mathscr{A})$ be a set equipped with an algebra. Suppose $T : Y \times Z \to W$ is bilinear and continuous. Suppose that $f \in \mathcal{R}_{\mathscr{A}}(X;Y)$ and $g \in \mathcal{R}_{\mathscr{A}}(X;Z)$ and let $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ be nets approaching f and g uniformly respectively. Then $T(f_\alpha, g_\beta)$ is $\mathscr{A}$-simple for each α, β and $\left(T(f_\alpha, g_\beta)\right)_{(\alpha,\beta)}$ approaches $T(f,g)$ uniformly. In particular, $T(f,g) \in \mathcal{R}_{\mathscr{A}}(X;W)$.

Proof. We first show that $T(f_\alpha, g_\beta)$ is $\mathscr{A}$-simple. Enumerate $f_\alpha[X] = \{y_1, \ldots, y_m\}$ and $g_\beta[X] = \{z_1, \ldots, z_n\}$. Then we see

$$T(f_\alpha, g_\beta)[X] = T\left[f_\alpha[X] \times g_\beta[X]\right] = \left\{T(y_j, z_k)\right\}_{1 \le j \le m, 1 \le k \le n}.$$

Then $T(f_\alpha, g_\beta)^{-1}[T(y_j, z_k)] = f_\alpha^{-1}[y_j] \cap g_\beta^{-1}[z_k] \in \mathscr{A}$. Then we claim that $\left(T(f_\alpha, g_\beta)\right)_{(\alpha,\beta)}$ is a net of simple functions approaching $T(f,g)$ uniformly. Let $U \subseteq W$ be an arbitrary neighborhood of the identity. Pick $V \subseteq U$ such that $V + V + V \subseteq U$. Pick $U_X \subseteq X$ and $U_Y \subseteq Y$ neighborhoods of the identity such that $U_X \times U_Y \subseteq T^{-1}[V]$. By boundedness of the images of f and g, pick $s, t > 1$ such that $sU_X \supseteq f[X]$ and $tU_Y \supseteq g[X]$. Finally, pick α', β' large enough such that for all $\alpha \ge \alpha'$ we have $f_\alpha(x) - f(x) \in \frac{1}{t} U_X$ for all $x \in X$ and for all $\beta \ge \beta'$ we have $g_\beta(x) - g(x) \in \frac{1}{s} U_Y$ for all $x \in X$. Then for all $(\alpha, \beta) \ge (\alpha', \beta')$, we find

$$T\left(f_\alpha(x), g_\beta(x)\right) - T\left(f(x), g(x)\right) = T\left(f_\alpha(x) - f(x), g_\beta(x) - g(x)\right) + T\left(f_\alpha(x) - f(x), g(x)\right) + T\left(f(x), g_\beta(x) - g(x)\right) \tag{6.1}$$

$$\in T\left[\frac{1}{s} U_X \times \frac{1}{t} U_Y\right] + T\left[\frac{1}{t} U_X \times t U_Y\right] + T\left[s U_X \times \frac{1}{s} U_Y\right] \tag{6.2}$$

$$\subseteq T[U_X \times U_Y] + T[U_X \times U_Y] + T[U_X \times U_Y] \subseteq V + V + V \subseteq U \tag{6.3}$$

where the first subset relation followed because $\frac{1}{s}, \frac{1}{t} \le 1$ and bilinearity of T. Our choice of U was arbitrary so we have the desired result. ∎

Corollary 6.16. Let $(X, \mathscr{A})$ be a set-algebra pair and Z be a $\mathcal{F}$-topological vector space. Let $f \in \mathcal{R}_{\mathscr{A}}(X;\mathbb{F})$ and $g \in \mathcal{R}_{\mathscr{A}}(X;Z)$. Then $fg \in \mathcal{R}_{\mathscr{A}}(X;Z)$.

6.2 Regulated functions on product spaces

We now turn our attention to product spaces. It is very natural to define the product of two sets equipped with algebras, as the next definition and proposition shows.

Definition 6.17. Suppose $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are two sets equipped with algebras. Then we define the product algebra $\mathscr{A} \otimes \mathscr{B}$ on $X \times Y$ as

$$\mathscr{A} \otimes \mathscr{B} = \left\{ \bigcup_{k=1}^{n} A_k \times B_k \;\middle|\; n \in \mathbb{N}, A_k \in A, B_k \in B \text{ for all } 1 \leq k \leq n \right\}.$$

Proposition 6.18. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be two sets equipped with algebras. Then $\mathscr{A} \otimes \mathscr{B}$ is indeed an algebra on $X \times Y$.

Proof. We check each of the elements of being an algebra in turn.

- We know that $\varnothing, X \in \mathscr{A}$ and $\varnothing, Y \in \mathscr{B}$, so we see that $\varnothing = \varnothing \times \varnothing \in \mathscr{A} \otimes \mathscr{B}$ and $X \times Y \in \mathscr{A} \otimes \mathscr{B}$.
- Let $E, F \in \mathscr{A} \otimes \mathscr{B}$ with $E = \bigcup\limits_{k=1}^{n} A_k \times B_k$ and $F = \bigcup\limits_{k=n+1}^{m} A_k \times B_k$. Then we see $E \cup F = \bigcup\limits_{k=1}^{m} A_k \times B_k \in \mathscr{A} \otimes \mathscr{B}$ as desired.
- Let $\bigcup\limits_{k=1}^{n} A_k \times B_k \in \mathscr{A} \otimes \mathscr{B}$ with $A_k \in \mathscr{A}$ and $B_k \in \mathscr{B}$ for all $1 \leq k \leq n$. Then we note that

$$\left(\bigcup_{k=1}^{n} A_k \times B_k \right)^{\mathsf{c}} = \bigcap_{k=1}^{n} (A_k \times B_k)^{\mathsf{c}} \tag{6.4}$$

$$= \bigcap_{k=1}^{n} \left(A_k^{\mathsf{c}} \times X \cup A_k \times B_k^{\mathsf{c}} \right) \tag{6.5}$$

$$= \bigcup_{b \in \{0,1\}^n} C_b \tag{6.6}$$

  where C_b is defined as follows: we first define $C_0^{(k)}$ as $A_k^{\mathsf{c}} \times X$ and $C_1^{(k)}$ as $A_k \times B_k^{\mathsf{c}}$. We then set $C_b = C_{b_1}^{(1)} \cap C_{b_2}^{(2)} \cap \cdots \cap C_{b_n}^{(n)}$, for $b = (b_1, \ldots, b_n) \in \{0,1\}^n$. However, now it is clear that $C_b \in \mathscr{A} \times \mathscr{B}$ as the intersection of products is just the product of the intersections and each $A_k, A_k^{\mathsf{c}} \in \mathscr{A}$ and $X, B_k^{\mathsf{c}} \in \mathscr{B}$. By the previous bullet, $\mathscr{A} \otimes \mathscr{B}$ is closed under unions so we have $\left(\bigcup\limits_{k=1}^{n} A_k \times B_k \right)^{\mathsf{c}} \in \mathscr{A} \otimes \mathscr{B}$.

Thus $\mathscr{A} \otimes \mathscr{B}$ contains $\varnothing$ and $X \times Y$ and is closed under unions and complements, so we have that $\mathscr{A} \otimes \mathscr{B}$ is an algebra as desired. ∎

We note that this must be the smallest algebra (ordered by inclusion) containing the rectangles, as any such algebra must necessarily contain finite unions of rectangles. In fact, we can even write each element of $\mathscr{A} \otimes \mathscr{B}$ as a disjoint union of rectangles.

Proposition 6.19. Let $E \in \mathscr{A} \otimes \mathscr{B}$. Then in fact E can be written as

$$E = \bigcup_{k=1}^{n} A_k \times B_k$$

where $A_k \in \mathscr{A}$, $B_k \in \mathscr{B}$, and $(A_k \times B_k) \cap (A_j \times B_j) = \varnothing$ for $k \neq j$.

Proof. Write $E = \bigcup_{k=1}^n A_k \times B_k$ where the union is not necessarily disjoint. We set $\mathcal{A} = \{\bigcap_{k\in F} A_k\}_{F\subseteq\{1,\dots,n\}} \subseteq \mathscr{A}$ and $\mathcal{B} = \{\bigcap_{k\in F} B_k\}_{F\subseteq\{1,\dots,n\}} \subseteq \mathscr{B}$, each of these ordered by inclusion. Enumerate the distinct nonempty inclusion-minimal elements of $\mathscr{A}$ as $A'_1,\dots,A'_m$, so the collection is pairwise disjoint. Similarly enumearte the distinct nonempty inclusion-minimal elements of $\mathscr{B}$ as $B'_1,\dots,B'_\ell$. We now note that for each $A'_p \times B'_q$, we have that either $A'_p \times B'_q \subseteq E$ or $A'_p \times B'_q \subseteq E^c$. Furthermore, it's obvious that

$$E \subseteq \bigcup_{\substack{1\le p\le m\\ 1\le q\le \ell}} A'_p \times B'_q.$$

Thus we can write

$$E = \bigcup_{\substack{1\le p\le m\\ 1\le q\le \ell}} E \cap (A'_p \times B'_q) = \bigcup_{\substack{1\le p\le m\\ 1\le q\le \ell\\ (A'_p\times B'_q)\subseteq E}} A'_p \times B'_q$$

We note that $A'_p \times B'_q \cap A'_r \times B'_s$ are disjoint when $(p,q) \neq (r,s)$ due to pairwise disjointness of the individual families. Thus we have the desired result. ∎

In Proposition 5.6, we saw that the topology of uniform convergence behaves nicely with respect to the currying map. In fact, this carries over to the regulated functions when we consider the product algebra.

Proposition 6.20. Let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be two sets equipped with algebras and let $(Z,\mathscr{U})$ be a uniform space. Define $T : \mathcal{F}(X;\mathcal{F}(Y;Z)) \to \mathcal{F}(X\times Y;Z)$ to be the natural currying map $Tf(x,y) = f(x)(y)$ as in Proposition 5.6. Then $T[\mathcal{R}_{\mathscr{A}}(X;\mathcal{R}_{\mathscr{B}}(Y;Z))] = \mathcal{R}_{\mathscr{A}\otimes\mathscr{B}}(X\times Y;Z)$.

Proof. By Proposition 6.14, we note $\mathcal{S}_{\mathscr{A}}(X;\mathcal{S}_{\mathscr{B}}(Y;Z))$ is dense in $\mathcal{R}_{\mathscr{A}}(X;\mathcal{R}_{\mathscr{B}}(Y;Z))$ and by construction we know that $\mathcal{S}_{\mathscr{A}\otimes\mathscr{B}}(X\times Y;Z)$ is dense in $\mathcal{R}_{\mathscr{A}\otimes\mathscr{B}}(X\times Y;Z)$, so it suffices to show that $T[\mathcal{S}_{\mathscr{A}}(X;\mathcal{S}_{\mathscr{B}}(Y;Z))] = \mathcal{S}_{\mathscr{A}\otimes\mathscr{B}}(X\times Y;Z)$ as T is a uniform homeomorphism. With that in mind, let $f \in \mathcal{S}_{\mathscr{A}}(X;\mathcal{S}_{\mathscr{B}}(Y;Z))$ and write

$$f = \sum_{k=1}^n s_k \chi_{A_k}$$

with $s_k = \sum\limits_{j=1}^{m_k} z_j^{(k)} \chi_{B_j} \in \mathcal{S}_{\mathscr{B}}(Y;Z)$. But then we see that

$$Tf = \sum_{k=1}^n \sum_{j=1}^{m_k} z_j^{(k)} \chi_{A_k\times B_j} \in \mathcal{S}_{\mathscr{A}\otimes\mathscr{B}}(X\times Y;Z).$$

On the other hand, suppose we have $f \in \mathcal{S}_{\mathscr{A}\otimes\mathscr{B}}(X \times Y; Z)$ with

$$f = \sum_{k=1}^{n} z_k \chi_{A_k \times B_k}.$$

Then we note that $z_k \chi_{B_k} \in \mathcal{S}_{\mathscr{B}}(Y;Z)$ and so

$$T^{-1} f = \sum_{k=1}^{n} \left(z_k \chi_{B_k}\right) \chi_{A_k} \in \mathcal{S}_{\mathscr{A}}\left(X; \mathcal{S}_{\mathscr{B}}(Y;Z)\right).$$

This shows the desired result. ■

6.3 Embeddings of Continuous Functions

Suppose that we have a set-algebra pair $(X, \mathscr{A})$ where X is also a topological space or uniform space. We can then ask when $C(X;Z)$ or $UC(X;Z)$ embeds into $\mathcal{R}_{\mathscr{A}}(X;Z)$. In the case where X is compact, a sufficient condition was given by Davison [1]. We rephrase his proof for our setup here.

Theorem 6.21 (Davison)**.** Suppose that (X, τ) is a compact topological space, $\mathscr{A}$ is an algebra on X, and $(Z, \mathscr{U})$ is a uniform space. Further suppose that $\mathscr{A} \cap X$ forms a base for τ. Then $C(X;Z)$ is a subset of $\mathcal{R}_{\mathscr{A}}(X;Z)$.

Proof. Let $f \in C(X;Z)$. We use the characterization in Proposition 6.6 to show that in fact f is $\mathscr{A}$-regulated. Let $U \in \mathscr{U}$ be arbitrary and pick symmetric $V \in \mathscr{U}$ such that $V^2 \subseteq U$. For each $x \in X$ pick some open set V_x such that $f[V_x] \subseteq U[f(x)]$. Pick $A_x \in \mathscr{A} \cap \tau$ such that $x \in A_x$ and $A_x \subseteq V_x$. We then pick a finite subcover $A_{x_1}, \ldots, A_{x_n}$ of X, and define $A_1, \ldots, A_n$ by inductively setting $A_1 = A_{x_1}$ and $A_k = A_{x_k} \setminus A_{k-1}$. Now we note for all $x, y \in A_k$, we have that $(f(x), f(x_k)) \in V$ and $(f(y), f(x_k)) \in V$, so $(f(x), f(y)) \in V^2 \subseteq U$. Thus f is $\mathscr{A}$-regulated as desired. ■

Theorem 6.22. Let (X, τ) be a locally compact topological space, $\mathscr{A}$ be an algebra on X, and let Z be a topological vector space. Further suppose that $\mathscr{A} \cap \tau$ forms a base for τ. Then $C_0(X;Z) \subseteq \mathcal{R}_{\mathscr{A}}(X;Z)$.

Proof. The exact same argument as before shows that any compactly supported function is $\mathscr{A}$-regulated. Then we note that $\mathcal{R}_{\mathscr{A}}(X;Z)$ is uniformly closed, so $C_0(X;Z) = \overline{C_c(X;Z)} \subseteq \mathcal{R}_{\mathscr{A}}(X;Z)$. ■

We know that all continuous function on a compact space are also uniformly continuous, in the unique uniformity on that compact space. Thus we can extend Davison's result to provide a sufficient condition for uniformly continuous functions.

Theorem 6.23. Let $(X,\mathscr{U}_X)$ and $(Z,\mathscr{U}_Z)$ be uniform spaces and let $\mathscr{A}$ be an algebra on X. Suppose that X is totally bounded and $\left\{\bigcup_{i=1}^n A_i \times A_i \;\middle|\; A_1,\dots,A_n \text{ forms an } \mathscr{A}\text{-partition of } X\right\} \cap \mathscr{U}_X$ forms a base of entourages for $\mathscr{U}_X$. Then $UC(X;Z) \subseteq \mathcal{R}_{\mathscr{A}}(X;Z)$.

Proof. Let $f \in UC(X;Z)$. Let $U \in \mathscr{U}_Z$ be arbitrary. By uniform continuity of fm there exists $V \in \mathscr{U}_X$ such that $(f \times f)[V] \subseteq U$. Pick a $\mathscr{A}$-partition $A_1,\dots,A_n$ of X such that $\bigcup_{i=1}^n A_i \times A_i \subseteq V$. But then $(f \times f)\left[\bigcup_{i=1}^n A_i \times A_i\right] \subseteq (f \times f)[V] \subseteq U$, which precisely tells us that $(f(x), f(y)) \in U$ whenever $x, y \in A_i$ for some i as desired. ■

7 The Cauchy Integral

We finally arrive at our goal - the Cauchy integral. We reference material on finitely additive measures which can be found in the appendix. The Cauchy integral on step functions will be defined in the obvious way with respect to a charge. However, we first need to show it's well-defined.

Lemma 7.1. Let $(X,\mathscr{A},\mu)$ be a signed charge space and let $s \in \mathcal{S}_{\mathscr{A}}(X;Z)$ with

$$s = \sum_{j=1}^{m} \alpha_j \chi_{A_j} = \sum_{k=1}^{n} \beta_k \chi_{B_k}.$$

Then $\sum\limits_{j=1}^{m} \mu(A_j)\alpha_j = \sum\limits_{k=1}^{n} \mu(B_k)\alpha_k$

Proof. The proof of this is routine and so is left as an exercise. ■

Definition 7.2. Let $(X,\mathscr{A},\mu)$ be a finite charge space and Z be a complete Hausdorff topological vector space. We then define the operator $\int_X \cdot \, \mathrm{d}\mu : \mathcal{S}_{\mathscr{A}}(X;Z) \to Z$ by setting for $s = \sum_{k=1}^m v_k \chi_{E_k}$

$$\int_X s \, \mathrm{d}\mu = \sum_{k=1}^{m} v_k \mu(E_k).$$

By virtue of Lemma 7.1 this operator is well-defined.

Now we can show that the integral is in fact linear and continuous when Z is locally convex.

Proposition 7.3. Suppose that $(X,\mathscr{A},\mu)$ is a finite charge space and Z is a complete Hausdorff locally convex vector space. Then $\int_X \cdot \, \mathrm{d}\mu : \mathcal{S}_{\mathscr{A}}(X;Z) \to Z$ is linear and continuous at 0 (and hence uniformly continuous).

Proof. Linearity follows by the same argument as for Lebesgue integration CITE. To show continuity at 0, let $U \subseteq Y$ be a convex neighborhood of 0. Pick $f \in \widehat{U}/\mu(X) \cap \mathcal{S}_{\mathscr{A}}(X;Y)$. We write $f = \sum_{k\leq m} y_k \chi_{E_k}$ in canonical form where the E_k partition X - in particular $y_k \in U/\mu(X)$. We then find $\int f \,\mathrm{d}\mu = \sum_{k\leq m} y_k \mu(E_k) = \mu(X) \sum_{k\leq m} y_k \frac{\mu(E_k)}{\mu(X)} \in U$ by convexity of U. This holds for all such f and our choice of U was arbitrary, so we have the desired result. ∎

Corollary 7.4. Suppose that $(X, \mathscr{A}, \mu)$ is a finite charge space and Z is a complete Hausdorff locally convex vector space. The operator $\int \cdot \,\mathrm{d}\mu$ extends uniquely to a continuous linear map $\int_X \cdot \,\mathrm{d}\mu : \mathcal{R}_{\mathscr{A}}(X;Z) \to Z$.

Proof. Since $\overline{\mathcal{S}_{\mathscr{A}}(X;Y)} = \mathcal{R}_{\mathscr{A}}(X;Y)$, $\int \cdot \,\mathrm{d}\mu$ is linear, and Z is complete, this follows by the standard theory of continuous extensions. ∎

In fact, local convexity is essential, as the next two results show.

Lemma 7.5. Given a topological vector space Y, the following are equivalent:

(a) Y is not locally convex.

(b) There is some open neighborhood U of 0 such that for all open 0-neighborhoods $W \subseteq U$, the convex hull of W, $\mathrm{co}(W)$ is not contained in U.

Proof. If (b) holds, then clearly Y is not locally convex. On the other hand, suppose (b) doesn't hold. Then for all open 0-neighborhoods U, there exists some open 0-neighborhood $W \subseteq U$ such that $\mathrm{co}(W) \subseteq U$. But then $\mathrm{co}(W)^\circ$ is an open convex neighborhood of 0 contained in U, so Y is locally convex as desired. ∎

Proposition 7.6. Let Y be a topological vector space. If Y is not locally convex, then the integral operator $\int \cdot \,\mathrm{d}\mu : \mathcal{S}_{\mathscr{B}([0,1])}([0,1];Y) \to Y$ is not continuous at 0.

Proof. Suppose Y is not locally convex. By the previous lemma, there is some open 0-neighborhood U such that for every open 0-neighborhood $W \subseteq U$, we know that $\mathrm{co}(W) \nsubseteq U$.
We claim that $I_\mu^{-1}[U]$ is not open. Towards a contradiction, suppose it was. Then we could find some open 0-neighborhood $W \subseteq Y$ such that $\tilde{W} \subseteq I_\mu^{-1}[U]$, as all such $\tilde{W}$-s form a neighborhood base at 0. Then $W \cap U$ is an open 0-neighborhood, so $\mathrm{co}(W \cap U) \nsubseteq U$. In particular, we can find $w_1, \ldots, w_n \in W \cap U$ and $t_1, \ldots, t_n \in [0,1]$ with $\sum_{i=1}^n t_i = 1$ and $\sum_{i=1}^n t_i w_i \notin U$. Let $A_1, \ldots, A_n \subseteq [0,1]$ be disjoint with $\bigcup_{i=1}^n A_i = [0,1]$ and $\lambda(A_i) = t_i$ for all $1 \leq i \leq n$. Define $f : [0,1] \to V$ piecewise by setting $f(x) = w_i$ if $x \in A_i$. Then clearly we see that $f \in \tilde{W}$. However, $I_\mu(f) = \sum_{i=1}^n \mu(A_i) w_i = \sum_{i=1}^n t_i w_i \notin U$, giving the desired contradiction. ∎

We can also define what it means to integrate over a subset of X.

Definition 7.7. Let $(X, \mathscr{A}, \mu)$ be a signed charge space, Z be a CHLCS and let $f \in \mathcal{R}_{\mathscr{A}}(X; Z)$. Suppose we have $A \in \mathscr{A}$. We then define

$$\int_A f \, \mathrm{d}\mu = \int f \chi_A \, \mathrm{d}\mu.$$

8 Properties of the Cauchy Integral

8.1 Basic Properties

Let us first show that the integral commutes with continuous linear operators.

Proposition 8.1. Let Y, Z be complete Hausdorff locally convex spaces and let $T : Y \to Z$ be linear and continuous. Then for any $f \in \mathcal{R}_{\mathscr{A}}(X; Y)$, we have that $T \circ f \in \mathcal{R}_{\mathscr{A}}(X; Z)$ and

$$\int_X T \circ f \, \mathrm{d}\mu = T\left(\int_X f \, \mathrm{d}\mu\right).$$

Proof. Since T is linear and continuous, we know that it is uniformly continuous. In particular, if we let $(s_\alpha)_\alpha$ be a net of $\mathscr{A}$-simple functions converging uniformly to f, then $T \circ s_\alpha$ is also $\mathscr{A}$-simple and $T \circ s_\alpha$ converges to $T \circ f$ uniformly by Lemma 6.12. From here, it is clear that $T \circ f \in \mathcal{R}_{\mathscr{A}}(X; Z)$.

To prove the integral identity, it suffices to prove that $\int_X T \circ s \, \mathrm{d}\mu = T\left(\int_X s \, \mathrm{d}\mu\right)$ for every simple function s - then the previous paragraph combined with taking limits gives the desired result. However, this is clear. Indeed, if we let $s = \sum_{k \le n} v_k \chi_{A_k}$ for $A_k \in \mathscr{A}$, we see that $T \circ s = \sum_{k \le n} T(v_k) \chi_{A_k}$ and so by linearity $\int T \circ s \, \mathrm{d}\mu = \sum_{k \le n} T(v_k) \mu(A_k) = T\left(\sum_{k \le n} v_k \mu(A_k)\right) = T\left(\int s \, \mathrm{d}\mu\right)$. ■

Proposition 8.2. Let Z a complete Hausdorff locally convex space and suppose $T : Z \to \mathbb{R}$ is sublinear and uniformly continuous. Then for any $f \in \mathcal{R}_{\mathscr{A}}(X; Y)$, we have that $T \circ f \in \mathcal{R}_{\mathscr{A}}(X; Z)$ and

$$T\left(\int f \, \mathrm{d}\mu\right) \le \int T \circ f \, \mathrm{d}\mu$$

Proof. This follows by an identical argument to the previous result ■

Proposition 8.3. Let $(X, \mathscr{A}, \mu)$ be a charge space and Z a complete Hausdorff locally convex space. Suppose $A, B \in \mathscr{A}$ are disjoint and $f \in \mathcal{R}_{\mathscr{A}}(X; Z)$. Then

$$\int_{A \cup B} f \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu + \int_B f \, \mathrm{d}\mu.$$

Proof. Let $(s_\alpha \chi_{A\cup B})_\alpha$ be a sequence of simple functions approaching $f\chi_{A\cup B}$. Then we note $(s_\alpha \chi_A)_\alpha$ approaches $f\chi_A$ uniformly and similarly for $(s_\alpha \chi_B)_\alpha$ and $f\chi_B$. But we see that $s_\alpha \chi_{A\cup B} = s_\alpha \chi_A + s_\alpha \chi_B$ and so the result follows by linearity and continuity of the integral. ■

8.2 The Dual of $\mathcal{R}_{\mathscr{A}}(X;\mathbb{R})$

We also have an elementary Riesz representation theorem for $\mathcal{R}_{\mathscr{A}}(X;\mathbb{R})$ and the Cauchy integral.

Theorem 8.4 (Riesz representation theorem)**.** Let $(X,\mathscr{A})$ be a set equipped with an algebra. Then we have the linear isomorphism $(\mathcal{R}_{\mathscr{A}}(X;\mathbb{R}))^* \simeq \kappa_{\mathscr{A}}(X)$.

Proof. By construction of the Cauchy integral, it's clear that every bounded measure in $\mathrm{ba}(\mathscr{A})$ defines a continuous linear map $\int \cdot \, \mathrm{d}\mu \in (\mathcal{R}_{\mathscr{A}}(X;\mathbb{R}))^*$. On the other hand, suppose we have $L \in (\mathcal{R}_{\mathscr{A}}(X;\mathbb{R}))^*$. Then define μ by letting $\mu(A) = L(\chi_A)$ for each $A \in \mathscr{A}$. To show our measure μ is of bounded total variation, it suffices to show μ has bounded range. Towards that end, we note that $|\mu(A)| \leq \|L\| \|\chi_A\| = \|L\|$. Furthermore, we see that $\int \chi_A \, \mathrm{d}\mu = \mu(A)$ so this truly is a correspondence. ■

In fact, we can even say something when the codomain is not $\mathbb{R}$.

Theorem 8.5. Let $(X,\mathscr{A})$ be a set-algebra pair and Z be a complete Hausdorff locally convex space. Then the map $B : \mathcal{R}_{\mathscr{A}}(X;Z) \times \kappa_{\mathscr{A}}(X) \to Z$ defined by $B(f,\mu) = \int_X f \, \mathrm{d}\mu$ is bilinear and continuous.

Proof. Let $\{[\cdot]_\alpha\}_\alpha$ be the seminorms defining the topology of Z. We note it suffices to show continuity at 0 and it suffices to show continuity on a subbasis, so pick some α and let $\varepsilon > 0$. Then we note that

$$\left[\int_X f \, \mathrm{d}\mu\right]_\alpha \leq \int_X [f(x)]_\alpha \, \mathrm{d}\mu(x) \leq [f]_\alpha \int_X 1 \, \mathrm{d}\mu(x) \leq [f]_\alpha \|\mu\|_\kappa.$$

Thus for $f \in \mathcal{R}_{\mathscr{A}}(X;Z)$ with $[f]_\alpha < \varepsilon$ and μ with $\|\mu\|_\kappa < \frac{1}{2}$, we have that $\left[\int_X f \, \mathrm{d}\mu\right]_\alpha \leq \frac{\varepsilon}{2} < \varepsilon$ which shows continuity as desired. ■

Let's examine a consequence of Theorem 8.4. Suppose that X is locally compact and $\mathcal{R}_{\mathscr{A}}(X;\mathbb{R})$ contains $C_0(X;\mathbb{R})$. Then every element of $\kappa_{\mathscr{A}}(X)$ defines a continuous linear functional on $C_0(X;\mathbb{R})$. However, we recall from the classical Riesz representation theorem that the dual of $C_0(X;\mathbb{R})$ are the Radon measures, and it is possible we have charges that are not measures. Thus we can have pure charges that act the same on continuous functions as a measure, but are not themselves measures!

Example 8.6. Define $\mathscr{A} \subseteq \mathcal{P}(\mathbb{R})$ by setting

$$\mathscr{A} = \left\{ \bigcup_{k=1}^{n} [a_k, b_k) \;\middle|\; a_k < b_k \text{ for all } 1 \le k \le n \right\} \cup \{\varnothing, \mathbb{R}\}.$$

It is easy to see that $C_0(\mathbb{R})$ is contained in $\mathcal{R}_{\mathscr{A}}(X;\mathbb{R})$. Fix $t \in \mathbb{R}$ and define $\mu_t(A) = 1$ if there exists a $\delta > 0$ such that $[t-\delta, t) \subseteq A$ and $\mu_t(A) = 0$ otherwise. One can verify then that $\|\mu\|_\kappa = 1$ and $\int_X f \, \mathrm{d}\mu_t = f(t)$ for each continuous function f, so μ_t acts the same as the measure δ_t. However, saliently, $\mu_t \neq \delta_t$!

8.3 Convexity of the Integral

Theorem 8.7 (Convexity of the integral)**.** Let $f \in \mathcal{R}_{\mathscr{A}}(X;Y)$. Then $\int_X f \, \mathrm{d}\mu \in \mu(X)\overline{\mathrm{co}}(f[X])$.

Proof. Pick a net of simple functions $(s_\alpha)_\alpha$ such that $s_\alpha \to f$ uniformly and $s_\alpha[X] \subseteq f[X]$ by Proposition 6.7. Then we note that $\int_X s_\alpha \, \mathrm{d}\mu \in \mu(X)\mathrm{co}(f[X])$ for each α, and so $\int_X f \, \mathrm{d}\mu = \lim_\alpha \int_X s_\alpha \, \mathrm{d}\mu \in \overline{\mathrm{co}}(f[X])$ as desired. ∎

We note that in the Bochner theory, the above theorem requires a more complicated Hahn-Banach argument to prove, but in this case it is perfectly elementary.

We now explore some consequences of convexity of the integral. To start, we'll show that the integral behaves nicely with respect to order.

Definition 8.8. We say $C \subseteq Z$ is a salient cone in Z if

(a) C is closed under nonnegative linear combinations, i.e. for all $s,t \ge 0$ and $c,d \in C$, we have that $sc + td \in C$ and

(b) $C \cap -C = \{0\}$.

For a salient cone C, we define the partial order $\le_C$ on Z by $y \le_C z \Leftrightarrow z - y \in C$ for $y,z \in Z$.

Corollary 8.9. Let $f,g \in \mathcal{R}_{\mathscr{A}}(X;Z)$ and let $C \subseteq Z$ be a closed salient cone in Z. Suppose $f(x) \ge_C g(x)$ for all $x \in X$. Then $\int_X f \, \mathrm{d}\mu \ge_C \int_X g \, \mathrm{d}\mu$.

Proof. By definition, we have that $g(x) - f(x) \in C$ for all $x \in X$, so $(f-g)[X] \subseteq C$. It's clear that C is convex and given that C is closed, so we have $\int_X g \, \mathrm{d}\mu - \int_X f \, \mathrm{d}\mu = \int_X g - f \, \mathrm{d}\mu \in \mu(X)C = C$. ∎

Corollary 8.10. Let $f,g \in \mathcal{R}_{\mathscr{A}}(X;\mathbb{R})$ and suppose that $f(x) \le g(x)$ for all $x \in X$. Then $\int_X f \, \mathrm{d}\mu \le \int_X g \, \mathrm{d}\mu$.

Proof. Apply the previous corollary to $g - f$ and use linearity of the integral. ■

In fact, in the real-valued case, we can say even more. We can restrict our simple functions to be bounded by f appropriately.

Proposition 8.11. Let $f \in \mathcal{R}_{\mathscr{A}}(X;\mathbb{R})$. Then

$$\int_X f \, \mathrm{d}\mu = \sup_{\substack{s \in \mathcal{S}_{\mathscr{A}}(X;\mathbb{R}) \\ s \leq f}} \int_X s \, \mathrm{d}\mu = \inf_{\substack{s \in \mathcal{S}_{\mathscr{A}}(X;\mathbb{R}) \\ s \geq f}} \int_X s \, \mathrm{d}\mu.$$

Proof. By Corolarry 8.9, one inequality is clear, so we will prove the other direction. In addition, we will only prove the first equality with the second following from an analogous argument. Pick a sequence of simple functions $(s_n)_n$ approaching f uniformly. Then we will define a new sequence of simple functions $(\tilde{s}_n)_n$ as follows: for each n, write

$$s_n = \sum_{k=1}^{m_n} \alpha_k^{(n)} \chi_{A_k^{(n)}}$$

for $A_k^{(n)} \in \mathscr{A}$. Define $\beta_k^{(n)} = \inf f \left[A_k^{(n)} \right]$ and set

$$\tilde{s}_n = \sum_{k=1}^{m_n} \beta_k^{(n)} \chi_{A_k^{(n)}}.$$

With this, it's obvious that $\tilde{s}_n \leq f$ for all n. We now claim that $\tilde{s}_n \to f$ uniformly. Towards proving this, let $\varepsilon > 0$. Pick N large enough such that for all $n \geq N$, we have that $|s_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all $x \in X$. We note by definition that for each $x \in A_k^{(n)}$, we have that $\left| f(x) - \alpha_k^{(n)} \right| < \frac{\varepsilon}{3}$, thus $\left| \beta_k^{(n)} - \alpha_k^{(n)} \right| = \left| \inf f \left[A_k^{(n)} \right] - \alpha_k^{(n)} \right| \leq \frac{\varepsilon}{3}$, so $|s_n(x) - \tilde{s}_n(x)| \leq \frac{\varepsilon}{2}$ for all $x \in X$ and so by the triangle inequality $|\tilde{s}_n(x) - f(x)| < \varepsilon$ for all $x \in X$. This holds for all such $n \geq N$, so the claim is proved. The result follows readily. ■

We now present an application of the ordering properties to show the Cauchy integral agrees with the Bochner integral on regulated functions.

Proposition 8.12. Let (X, Σ, μ) be a measure space and let Z be a Banach space. Suppose $f \in \mathcal{R}_{\mathscr{A}}(X;Z)$. Then the Bochner integral of f, $B\int_X f \, \mathrm{d}\mu$ equals the Cauchy integral $\int_X f \, \mathrm{d}\mu$.

Proof. The Bochner and Cauchy integral clearly agree on step functions as they are defined the same way. Pick a sequence of step functions $(s_n)_n$ approaching f uniformly. In particular, they approach f pointwise and so f is Bochner measurable. We then note that $\|f - s_n\| \le \sup_{x\in X}\|f(x) - s_n(x)\|$, so

$$\int_X \|f - s_n\| \, \mathrm{d}\mu \le \sup_{x\in X}\|f(x) - s_n(x)\| \int_X 1 \, \mathrm{d}\mu = \sup_{x\in X}\|f(x) - s_n(x)\|\mu(X).$$

The right hand side goes to zero, and so the left hand side does as well which then implies that $B\int_X f \, \mathrm{d}\mu = \lim_n \int_X s \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu$ as desired. ∎

Another consequence of convexity is absolute continuity of the integral.

Theorem 8.13 (Absolute continuity of the integral)**.** Let $f \in \mathcal{R}_{\mathscr{A}}(X;Y)$. Then for all 0-neighborhoods $U \subseteq Y$, there exists $\delta > 0$ such that for all $A \in \mathscr{A}$ with $\mu(A) < \delta$, we have that

$$\int_A f \, \mathrm{d}\mu \in U.$$

Proof. Fix a 0-neighborhood $U \subseteq Y$. Pick a closed convex 0-neighborhood V such that $V \subseteq U$. We know that $f[X]$ is bounded, so pick $t > 0$ such that $f[X] \subseteq tV$ and set $\delta = \frac{1}{t}$. Then for all $A \in \mathscr{A}$ with $\mu(A) < \delta$, we see by convexity of the integral that

$$\int_A f \, \mathrm{d}\mu \in \mu(A)\overline{\mathrm{co}}(f[X]) \subseteq \mu(A)\overline{\mathrm{co}}(tV) = \mu(A)tV \subseteq V \subseteq U$$

as desired. ∎

8.4 Fubini's Theorem

We now present a simple version of Fubini's theorem for the Cauchy integral. In fact, due to the fact our integral can easily handle vector-valued functions, Fubini is just a simple consequence of Proposition 6.20.

Theorem 8.14 (Fubini)**.** Let $f \in \mathcal{R}_{\mathscr{A}\times\mathscr{B}}(X \times Y; Z)$. Then

$$\int_{X\times Y} f \, \mathrm{d}(\mu \times \nu) = \int_X \int_Y f \, \mathrm{d}\nu \, \mathrm{d}\mu$$

Proof. Pick a net of simple functions $(f_\alpha)_\alpha \subseteq \mathcal{S}_{\mathscr{A}\times B}(X \times Y; Z)$ such that $f_\alpha \to f$ uniformly, using Lemma ??? to write it as

$$f_\alpha = \sum_{k=1}^{n_\alpha} z_k^{(\alpha)} \chi_{A_k^{(\alpha)} \times B_k^{(\alpha)}}$$

for $z_k^{(\alpha)} \in Z$, $A_k^{(\alpha)} \in \mathscr{A}$, and $B_k^{(\alpha)} \in \mathscr{B}$ for each k, α and $A_k^{(\alpha)} \times B_k^{(\alpha)} \cap A_j^{(\alpha)} \times B_j^{(\alpha)} = \varnothing$ for $j \neq k$. By definition of the product charge space, we have

$$\int_{X\times Y} f_\alpha \, \mathrm{d}(\mu\times\nu) = \sum_{k=1}^{n_\alpha} z_k^{(\alpha)} \mu\left(A_k^{(\alpha)}\right) \nu\left(B_k^{(\alpha)}\right)$$

We note that we can in fact consider f_α and f as functions from $X \to S_{\mathscr{A}}(Y;Z)$ in the obvious way, and by Proposition 6.20 we know that $f_\alpha \to f$ in $\mathcal{R}_{\mathscr{A}}(X;\mathcal{R}_{\mathscr{B}}(Y;Z))$. We then have

$$f_\alpha(x) = \sum_{k=1}^{n_\alpha} \left(z_k^{(\alpha)} \chi_{B_k^{(\alpha)}}\right) \chi_{A_k^{(\alpha)}}(x)$$

for each $x \in X$. In particular, this implies

$$\int_X f_\alpha \, \mathrm{d}\mu = \sum_{k=1}^{n_\alpha} \left(z_k^{(\alpha)} \chi_{B_k^{(\alpha)}}\right) \mu\left(A_k^{(\alpha)}\right).$$

We note that $\lim_\alpha \int_X f_\alpha \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu$ where the limit is taken uniformly (in $\mathcal{R}_{\mathscr{B}}(Y;Z)$). Integrating the above again with respect to ν, we find

$$\int_Y \int_X f_\alpha \, \mathrm{d}\mu \, \mathrm{d}\nu = \sum_{k=1}^{n_\alpha} z_k^{(\alpha)} \mu\left(A_k^{(\alpha)}\right) \nu\left(B_k^{(\alpha)}\right) = \int_{X\times Y} f_\alpha \, \mathrm{d}(\mu\times\nu)$$

Now taking limits on both sides, we obtain the desired result. ■

8.5 Change of Variables

Lemma 8.15. Let $U \subseteq \mathbb{R}^n$ be open, $E \subseteq U$ be Peano-Jordan measurable, and $\varphi : U \to \mathbb{R}^n$ be Lipschitz and a homeomorphism onto its image. Then $\varphi[E]$ is also Peano-Jordan measurable. [7]

Lemma 8.16. Let $f \in \mathcal{R}_{\mathscr{A}_n}(\mathbb{R}^n;Z)$ and φ be a homeomorphism onto its image with φ^{-1} Lipschitz. Then $f \circ \varphi \in \mathcal{R}_{\mathscr{A}_n}(\mathbb{R}^n;Z)$.

Lemma 8.17. Let $E \subseteq \mathbb{R}^n$ be Peano-Jordan measurable and $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible. Then $\mu(T[E]) = |\det T|\mu(E)$.

Proposition 8.18 (Linear change of variables)**.** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible and $U \subseteq \mathbb{R}^n$ be bounded and open. Also, let $f : U \to Y$ be regulated with $\overline{\operatorname{supp} f} \subseteq U$. Then $f \circ T$ is also regulated and

$$\int_{T[U]} f \, \mathrm{d}\mu = |\det T| \int_U f \circ T \, \mathrm{d}\mu$$

Proof. Let $(s_\alpha)_\alpha$ be a net of simple functions approximating f uniformly. Then we note that for any $E \subseteq \mathbb{R}^n$ that is Peano-Jordan measurable, we have that $\chi_E \circ T = \chi_{T^{-1}E}$. By Lemma 8.15, $T^{-1}E$ is measurable and so $s_\alpha \circ T$ is also a simple function. To show $f \circ T$ is regulated, it suffices to show $s_\alpha \circ T \to f \circ T$ uniformly. Towards that end, let $V \subseteq Y$ be a neighborhood of the identity. Pick β such that for all $\alpha \geq \beta$ and all $z \in T[U]$, we have $s_\alpha(z) - f(z) \in V$. In particular, then for $x \in U$, we know that $T(x) \in T[U]$ and so $s_\alpha(T(x)) - f(T(x)) \in V$. Since V was arbitrary, we have $s_\alpha \circ T \to f \circ T$ uniformly as desired.

Let us now prove the integral identity. Lemma 8.17 tells us it holds for characteristic functions, since

$$\int_{T[U]} \chi_{T[E]} \, \mathrm{d}\mu = \mu(T[E]) = |\det T|\mu(E) = |\det T| \int_U \chi_E \, \mathrm{d}\mu$$

for any Peano-Jordan measurable $E \subseteq U$. From here, linearity grants us simple functions and continuity of the integral combined with the previous paragraph gives the desired result. ∎

Our goal will be to prove the previous theorem for diffeomorphisms in the place of linear maps. From now on, we fix bounded open sets $U, V \subseteq \mathbb{R}^n$ and $\varphi : U \to V$ a C^1 diffeomorphism.

Lemma 8.19. For any Peano-Jordan measurable set E with $\overline{E} \subseteq U$, we have

$$\mu(\varphi(E)) \leq \int_E |\det D\varphi| \, \mathrm{d}\mu$$

Proof. We start by analyzing cubes. Indeed, let $Q \subseteq U$ be any cube whose closure lies in U with center c and side length s. The mean value theorem implies for any $\psi : U \to \mathbb{R}^n$ continuously differentiable and $y \in Q$, we have

$$\left\|\psi(y) - \psi(c)\right\|_\infty \leq \left\|y - c\right\|_\infty \sup_{z \in Q} \|D\varphi(z)\|.$$

If we let P be the cube centered at $\psi(c)$ with side length $s \sup_{z \in Q} \|D\varphi(z)\|$, the above implies $\psi[Q] \subseteq P$ and so

$$\mu(\psi[Q]) \leq \mu(P) = \left(s \sup_{z \in Q} \|D\psi(z)\|\right)^n = \left(\sup_{z \in Q} \|D\psi(z)\|\right)^n \mu(Q). \tag{$*$}$$

We continue with the proof. Let $\varepsilon > 0$. $D\varphi$ is compactly supported, so by uniform continuity pick δ small enough such that $\left\|D\varphi(x)^{-1} D\varphi(y)\right\| < 1 + \varepsilon$ and $\left|\det D\varphi(y)\right| > (1 - \varepsilon)|\det D\varphi(x)|$ whenever $\left\|x - y\right\|_\infty < \delta$. Pick a cubic cover $\{Q_k\}_{k \leq n}$ of E such that each cube has side length less than δ, and for each k pick some $x_k \in Q_k$.

Setting $\psi = D\varphi(x_k)^{-1} \circ \varphi$, we see that

$$\begin{aligned}
\mu(\varphi[E]) &\leq \mu\left(\varphi\left[\bigcup_{k\leq n} Q_k\right]\right) \\
&\leq \sum_{k\leq n} \mu(\varphi[Q_k]) \\
&= \sum_{k\leq n} |\det D\varphi(x_k)|\mu(\psi[Q_k]) && \text{(Lemma 8.17)} \\
&\leq \sum_{k\leq n} |\det D\varphi(x_k)| \left(\sup_{z\in Q} \left\| D\varphi(x_k)^{-1} D\varphi(z)\right\|\right)^n \mu(Q_k) \\
&\leq \sum_{k\leq n} (1+\varepsilon)^n |\det D\varphi(x_k)|\mu(Q_k) \\
&\leq \sum_{k\leq n} (1+\varepsilon)^n (1-\varepsilon)^{-1} \int_{Q_k} \left|\det D\varphi(y)\right| \mathrm{d}y \\
&\leq (1+\varepsilon)^n (1-\varepsilon)^{-1} \int_{\bigcup_{k\leq n} Q_k} \left|\det D\varphi(y)\right| \mathrm{d}y.
\end{aligned}$$

Since ε and our cubic covering was arbitrary, continuity of the integral gives the desired result. ■

Lemma 8.20. Let $U \subseteq \mathbb{R}^n$ be open and $\varphi : U \to \varphi[U]$ be a C^1 diffeomorphism. Let $f \in \mathcal{R}_{\mathscr{A}}(\mathbb{R}^n; \mathbb{R})$ be such that $\operatorname{supp}(f) \subseteq \varphi[U]$ and $f \geq 0$. Then

$$\int_{\varphi[U]} f \, \mathrm{d}\mu \leq \int_U (f \circ \varphi)|\det D\varphi| \, \mathrm{d}\mu.$$

Proof. Let $s \in \mathcal{S}_{\mathscr{A}}(\mathbb{R}^n; \mathbb{R})$ be a simple function with $0 \leq s \leq f$. Then

$$\begin{aligned}
\int_{\varphi[U]} s \, \mathrm{d}\mu &= \sum_{k=1}^{n} s(x_i)\mu(\varphi[A_i]) \\
&\leq \sum_{k=1}^{n} s(x_i) \int_{A_i} |\det D\varphi| \, \mathrm{d}\mu \\
&= \sum_{k=1}^{n} \int_{A_i} (s \circ \varphi)|\det D\varphi| \, \mathrm{d}\mu \\
&\leq \sum_{k=1}^{n} \int_{A_i} (f \circ \varphi)|\det D\varphi| \, \mathrm{d}\mu \leq \int_U (f \circ \varphi)|\det D\varphi| \, \mathrm{d}\mu
\end{aligned}$$

■

Theorem 8.21 (Change of variables)**.** Let $U \subseteq \mathbb{R}^n$ be open and $\varphi : U \to \varphi[U]$ be a C^1 diffeomorphism. Let

Z be a complete locally convex space. Let $f \in \mathcal{R}_{\mathscr{A}}(\mathbb{R}^n; Z)$ be such that $\operatorname{supp}(f) \subseteq \varphi[U]$. Then

$$\int_{\varphi[U]} f \, \mathrm{d}\mu = \int_U (f \circ \varphi) |\det D\varphi| \, \mathrm{d}\mu.$$

Proof. We first prove this when $f \geq 0$. Indeed, applying the previous theorem to $(f \circ \varphi)|\det D\varphi|$ and φ^{-1} on $\varphi[U]$, we see

$$\int_{\varphi^{-1}[\varphi[U]]} (f \circ \varphi)|\det D\varphi| \, \mathrm{d}\mu \leq \int_{\varphi[U]} (f \circ \varphi \circ \varphi^{-1}) \left|\det D\varphi\right| \left|\det D\varphi^{-1}\right| = \int_{\varphi[U]} f \, \mathrm{d}\mu$$

which shows $\int_{\varphi[U]} f \, \mathrm{d}\mu = \int_U (f \circ \varphi)|\det D\varphi| \, \mathrm{d}\mu$. In particular, this gives $\mu(\varphi[E]) = \int_E |\det D\varphi| \, \mathrm{d}\mu$.

We now prove this for any generic function. Let $f \in \mathcal{R}_{\mathscr{A}}(\mathbb{R}^n; Z)$. Then pick a net of $\mathscr{A}$-simple functions $(s_\alpha)_\alpha$ converging to f uniformly. We then note that $\left((s_\alpha \circ \varphi)|\det D\varphi|\right)_\alpha$ converges uniformly to $(f \circ \varphi)|\det D\varphi|$ which is also an $\mathscr{A}$-simple function. Thus we see

$$\int_{\varphi[U]} f \, \mathrm{d}\mu = \lim_\alpha \int_{\varphi[U]} s_\alpha \, \mathrm{d}\mu = \lim_\alpha \sum_{k=1}^{n_\alpha} z_k^{(\alpha)} \mu(\varphi[A_k^{(\alpha)}]) = \lim_\alpha \sum_{k=1}^{n_\alpha} z_k^{(\alpha)} \int_{A_k^{(\alpha)}} |\det D\varphi| \, \mathrm{d}\mu = \int_U (f \circ \varphi)|\det D\varphi|$$

∎

8.6 Fundamental Theorem of Calculus

We note that convexity of the integral functions as a sort of mean value theorem for us, and so we can prove the fundamental theorems of calculus.

Theorem 8.22 (First fundamental theorem of calculus)**.** Let Z be a locally convex space and $a, b \in \mathbb{R}$ with $a < b$. Suppose that $f \in C((a,b); Z)$ and define $F : (a,b) \to Z$ by

$$F(x) = \int_a^x f \, \mathrm{d}\mu.$$

Then F is continuous and Gateaux differentiable with $F'(x) = f(x)$ for all $x \in (a,b)$.

Proof. Continuity of the integral and additivity with respect to sets readily shows that F is continuous. To show differentiability, let U be a closed convex neighborhood of the identity. Let $t > 0$ be such that for all $0 < h < t$, we have $f(y) - f(x) \in U$ for all $y \in [x, x+h]$. We then note

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f \, \mathrm{d}\mu$$

using additivity with respect to sets. By convexity of the integral, we have

$$\frac{1}{h}\int_x^{x+h} f \,\mathrm{d}\mu \in \overline{\mathrm{co}}\Big(f\Big[[x,x+h]\Big]\Big) \subseteq f(x) + \overline{\mathrm{co}}(U) = f(x) + U.$$

This holds for all $0 < h < t$, and since our choice of U was arbitrary, we conclude that $\frac{F(x+h)-F(x)}{h} \to f(x)$ as $h \to 0$ as desired. ■

In fact, [4] shows there is also a mean value inequality in locally convex spaces, allowing us to prove the following lemma and the second fundamental theorem of calculus.

Lemma 8.23. Let Z be a complete Hausdorff locally convex space. Suppose $f : (a,b) \to Z$ is continuous and Gateaux differentiable with $f'(x) = 0$ for all $x \in (a,b)$. Then f is constant.

Proof. Let $([\cdot]_\alpha)_\alpha$ be seminorms determining the topology of Z. Let $x_0 \in (a,b)$. By the mean value inequality for seminorms [4], we see that $[f(x) - f(x_0)]_\alpha \leq [f'(y)]_\alpha = 0$ for some y between x and x_0. This holds for each α, so we conclude that $f(x) - f(x_0) = 0$ as desired. ■

Theorem 8.24 (Second fundamental theorem of calculus)**.** Let $a < b$ and Z be a complete Hausdorff locally convex space. Suppose that f is Gateaux differentiable on (a,b) with $f' \in C((a,b);Z)$. Then

$$f(x) - f(a) = \int_a^x f' \,\mathrm{d}\mu.$$

Proof. Define $F : (a,b) \to Z$ as $F(x) = \int_a^x f' \,\mathrm{d}\mu$. The first fundamental theorem of calculus tells us that F' is differentiable with $F' = f'$ on (a,b). Then $(F - f)' = 0$ and thus is constant by Lemma 8.23. In particular, we see that

$$f(x) - f(a) = F(x) - F(a) = \int_a^x f' \,\mathrm{d}\mu - \int_a^a f' \,\mathrm{d}\mu = \int_a^x f' \,\mathrm{d}\mu$$

as desired. ■

Part III

Appendices

A Some facts about uniformities

The appropriate level of generality we work with for the Cauchy integral is that of *uniformities*. We set some notation and recall some facts about uniformities. These facts can be found in General Topology by John Kelley, so we state them without proof.

Definition A.1. Let Z be a set. We define the following operations on $Z \times Z$

(a) For $V \subseteq Z \times Z$, we define the *inverse* of V as

$$V^{-1} = \{(z, z') \in Z \times Z \mid (z', z) \in V\}.$$

It is clear that inversion is an involution on $\mathcal{P}(Z \times Z)$.

(b) For $V, W \subseteq Z \times Z$, we define the *composition* of V and W

$$V \circ W = \{(y, z) \in Z \times Z \mid \text{there exists } w \in Z \text{ such that } (y, w) \in V \text{ and } (w, z) \in W\}.$$

It is clear that composition is associative. Sometimes, we will omit the infix and just write $VW = V \circ W$ and we write V^n to mean $\underbrace{V \circ \cdots \circ V}_{n \text{ times}}$.

A *uniformity* on Z is a set $\mathscr{U} \subseteq \mathcal{P}(Z \times Z)$ such that

(a) $\mathscr{U} \neq \varnothing$.

(b) If $U, V \in \mathscr{U}$, then $U \cap V \in \mathscr{U}$.

(c) If $U \in \mathscr{U}$ and $U \subseteq V \subseteq Z \times Z$, then $V \in \mathscr{U}$.

(d) For all $U \in \mathscr{U}$, we have that $\Delta_Z \subseteq U$ where $\Delta_Z = \{(z, z) \mid y \in Z\}$.

(e) If $U \in \mathscr{U}$, then $U^{-1} \in \mathscr{U}$.

(f) If $U \in \mathscr{U}$, then there exists $V \in \mathscr{U}$ such that $V^2 \subseteq U$.

In this case, we call $(Z,\mathscr{U})$ a *uniform space* and elements of $\mathscr{U}$ are called *entourages* of Z.

From now on we fix a uniform space $(Z,\mathscr{U})$.

Definition A.2. We say that $\mathscr{B} \subseteq \mathscr{U}$ is a *base of entourages* for $\mathscr{U}$ if for every entourage $U \in \mathscr{U}$ there is $B \in \mathscr{B}$ such that $B \subseteq U$.

Proposition A.3. Suppose $\mathscr{B} \subseteq \mathcal{P}(Z \times Z)$ satisfies

(a) $\mathscr{B} \neq \varnothing$.

(b) For every $U, V \in \mathscr{B}$, there is $W \in \mathscr{B}$ with $W \subseteq U \cap V$.

(c) For all $U \in \mathscr{B}$, $\Delta_Z \subseteq U$.

(d) For every $U \in \mathscr{B}$, there is $V \in \mathscr{B}$ with $V \subseteq U^{-1}$.

(e) For every $U \in \mathscr{B}$, there is $V \in \mathscr{B}$ with $V^2 \subseteq U$.

Then $\mathscr{U} = \{U \subseteq Z \times Z \mid \text{ there exists } V \in \mathscr{B} \text{ such that } V \subseteq U\}$ is a uniformity on Z and $\mathscr{B}$ forms a fundamental system of entourages for $\mathscr{U}$.

Examples A.4. Uniform spaces generalize two other common structures - metric spaces and topological groups.

(a) Let (X,d) be a extended pseudometric space. Then for $\varepsilon > 0$, define the set

$$U_\varepsilon = \{(x,y) \in X \;\big|\; d(x,y) < \varepsilon\}.$$

The sets $\{U_\varepsilon\}_{\varepsilon>0}$ satisfies the conditions of Proposition A.3 and so define a uniformity on X.

(b) Let G be a topological group. Then for each neighborhood of the identity $V \subseteq G$, define

$$V_L = \left\{(x,y) \in X \;\big|\; x^{-1}y \in V\right\}.$$

Then V_L satisfies the conditions of Proposition A.3 and so defines a uniformity on G, usually called the *left uniformity*. In a similar fashion one can define a right uniformity (replacing $x^{-1}y$ by xy^{-1}) on G.

From now on, if we ever have a metric space or topological group, we will equip them with the suitable uniformity as defined above.

Definition A.5. Let $(Z,\mathscr{U})$ be a uniform space. For $U \in \mathscr{U}$ and $E \subseteq Z$, we define

$$U[E] = \{z \in Z \mid (w,z) \in U \text{ for some } w \in E\}.$$

In addition, for $z \in Z$, we set $U[z] = U[\{z\}]$.

Proposition A.6. Let $(Z,\mathscr{U})$ be a uniform space. The set

$$\{V \subseteq Z \mid \text{for all } z \in Z \text{ there exists } U \in \mathscr{U} \text{ such that } U[z] \subseteq V\}$$

is a topology on Z. We call this the *topology induced by* $\mathscr{U}$.

Proposition A.7. Let $(Z,\mathscr{U})$ be a uniform space and $E \subseteq Z$. Then the set

$$\mathscr{U}_E = \{U \subseteq E \times E \mid \text{there exists } V \in \mathscr{U} \text{ such that } U = V \cap (E \times E)\}$$

is a uniformity on E, and the topology induced by $\mathscr{U}_E$ is precisely the subspace topology from the topology induced by $\mathscr{U}$. We call $\mathscr{U}_E$ the *subspace uniformity on* E.

Definition A.8. Let $(Z,\mathscr{U})$ be a uniform space.We say a set $E \subseteq Z$ is $\mathscr{U}$*-bounded* if for every $U \in \mathscr{U}$, there exists a finite set $F \subseteq E$ and $m \in \mathbb{N}$ such that $E \subseteq U^m[F]$.

Remark A.9. It is worth noting that $\mathscr{U}$-boundedness is not necessarily equivalent to metric boundedness. Indeed, consider $\mathbb{R}$ with the metric $d(x,y) = \frac{|x-y|}{1+|x-y|}$. One can check that the identity map from $(\mathbb{R}, |\cdot|)$ to $(\mathbb{R}, d)$ is a uniform homeomorphism, but in one case $\mathbb{R}$ itself is metric bounded, but in the other it is metric unbounded.

Definition A.10. Let $(Z,\mathscr{U})$ be a uniform space. We say a set $E \subseteq Z$ is *totally bounded* if for every $U \in \mathscr{U}$, there exists a finite set $F \subseteq E$ such that $E \subseteq U[F]$.

Definition A.11. Let $(Z,\mathscr{U})$ be a uniform space. We say a net $(z_\alpha)_\alpha$ in Z is *Cauchy* if for every $U \in \mathscr{U}$ there exists some γ such that for all $\alpha, \beta \geq \gamma$, we have that $(y_\alpha, y_\beta) \in U$. We say Z is *complete* if every Cauchy net is convergent (in the topology induced by $\mathscr{U}$).

Definition A.12. Suppose that $(X,\mathscr{U}_X)$ and $(Z,\mathscr{U}_Z)$ are uniform spaces. We say that a map $f : X \to Z$ is *uniformly continuous* if for every $U \in \mathscr{U}_Z$ there is some $V \in \mathscr{U}_X$ such that for all $(x,y) \in V$ we know that $(f(x), f(y)) \in U$.

Proposition A.13. Let $(X, \mathscr{U}_X)$ and $(Z, \mathscr{U}_Z)$ be uniform spaces and $f : X \to Z$. Then the following are equivalent:

(a) f is uniformly continuous.

(b) There exist bases of entourages $\mathscr{B}_X$ for $\mathscr{U}_X$ and $\mathscr{B}_Z$ for $\mathscr{U}_Z$ such that for every $V \in \mathscr{B}_Z$ there exists a $U \in \mathscr{B}_X$ such that $U \subseteq (f \times f)^{-1}[V]$

Theorem A.14. Suppose that $(X, \mathscr{U}_X)$ is a uniform space and $(Z, \mathscr{U}_Z)$ is a complete uniform space, and suppose $D \subseteq X$ is dense (in the topology induced by $\mathscr{U}$). Further suppose that $f : D \to Z$ is uniformly continuous (with respect to the subspace uniformity on D). Then there exist a unique uniformly continuous function $\tilde{f} : X \to Z$ such that $\tilde{f}\restriction_D = f$.

A.1 Examples

We now present two common and useful examples of uniformities on function spaces.

Example A.15. Let X be a set and $(Z, \mathscr{U})$ be a uniform space. We wish to uniformize pointwise convergence. Towards that end, we define the set

$$B_{U,F} = \left\{(f, g) \in Z^X \times Z^X \;\middle|\; (f(x), g(x)) \in U \text{ for all } x \in F\right\}.$$

for every entourage $U \in \mathscr{U}$ and finite set $F \subseteq X$. One can check that the sets $\{B_{U,F} \;\big|\; U \in \mathscr{U}, F \subseteq X \text{ is finite}\}$ satisfies the conditions of Proposition A.3 and so defines a base for a uniformity on Z^X, called the *uniformity of pointwise convergence*. It is easy to check that a net $(f_\alpha)_\alpha$ of functions from X to Z converges to some $f : X \to Z$ if and only if $f_\alpha(x) \to f(x)$ for each $x \in X$.

Example A.16. Let X be a set and $(Z, \mathscr{U})$ be a uniform space. In similar fashion to the previous example, we can also uniformize uniform convergence. We define the set

$$B_U = \left\{(f, g) \in Z^X \times Z^X \;\middle|\; (f(x), g(x)) \in U \text{ for all } x \in X\right\}.$$

Then again the set $\{B_U\}_{U \in \mathscr{U}}$ satisfies the conditions of Proposition A.3 and so defines a base for a uniformity on Z^X, called the *uniformity of uniform convergence*. In this thesis, we set $\mathcal{F}(X;Z)$ to be the set Z^X endowed with the uniformity of uniform convergence.

B Some facts about charges

Definition B.1. Let $(X, \mathscr{A})$ be a set-algebra pair. Then we say $\mu : \mathscr{A} \to [0, \infty)$ is a (bounded real) *charge* on $(X, \mathscr{A})$ if it satisfies the following:

(a) $\mu(\emptyset) = 0$.

(b) For any $A, B \in \mathscr{A}$ with $A \cap B = \emptyset$, we have that $\mu(A \cup B) = \mu(A) + \mu(B)$

B.1 Peano-Jordan Completion

We define the Peano-Jordan completion as in [7] and prove some results about it.

Definition B.2. Let $(X, \mathscr{A}, \mu)$ be a charge space. We define the inner and outer charges associated to μ

$$\mu_i(A) = \sup\{\mu(E) \mid E \subseteq A\}$$

$$\mu_o(A) = \inf\{\mu(E) \mid A \subseteq E\}$$

for any $A \in \mathcal{P}(X)$. In addition, we set

$$\mathscr{A}^* = \{A \in \mathcal{P}(X) \mid \mu_i(A) = \mu_o(A)\}.$$

In addition, we define $\mu^* : \mathscr{A}^* \to [0, \infty]$ by setting $\mu^*(A) = \mu_o(A) = \mu_i(A)$ for $A \in \mathscr{A}^*$.

From the definition, we see μ^* extends μ - if $A \in \mathscr{A}$, then clearly $\mu_i(A) = \mu_o(A) = \mu(A)$.

Lemma B.3. $A^* \in \mathscr{A}^*$ if and only if for every $\varepsilon > 0$ there exists $E, F \in \mathscr{A}$ with $E \subseteq A^* \subseteq F$ and $\mu(F \setminus E) < \varepsilon$.

Proof. Let $A^* \in \mathscr{A}^*$ and pick $E \subseteq A^* \subseteq F$ such that $\mu(A^*) - \mu(E) < \frac{\varepsilon}{2}$ and $\mu(F) - \mu(A^*) < \frac{\varepsilon}{2}$. Then $\mu(F \setminus E) = \mu(F) - \mu(E) = (\mu(F) - \mu(A^*)) + (\mu(A^*) - \mu(E)) < \varepsilon$ which shows what we wanted.
On the other hand, suppose the second condition holds for A. Then we see for every $\varepsilon > 0$, we can find $E \subseteq A \subseteq F$ with $\mu(F \setminus E) < \varepsilon$. In particular, we have $\mu(E) \leq \mu_i(A) \leq \mu_o(A) \leq \mu(F)$ and so $\mu_o(A) - \mu_i(A) \leq \mu(F) - \mu(E) < \varepsilon$. This holds for every $\varepsilon > 0$, so we get $\mu_o(A) = \mu_i(A)$, i.e. that $A \in \mathscr{A}^*$. ∎

Proposition B.4. $\mathscr{A}^*$ is an algebra.

Proof. We check each element of being an algebra in turn.

(a) We note that $\emptyset, X \in \mathscr{A}$, and clearly $\mathscr{A}^*$ contains $\mathscr{A}$, so $\emptyset, X \in \mathscr{A}$.

(b) Let $A, B \in \mathscr{A}^*$ be disjoint and let $\varepsilon > 0$. Pick $A_0 \subseteq A \subseteq A_1$ and $B_0 \subseteq B \subseteq B_1$ such that $\mu(A_1 \setminus A_0) < \frac{\varepsilon}{2}$ and $\mu(B_1 \setminus B_0) < \frac{\varepsilon}{2}$. Then we have $A_0 \cup B_0 \subseteq A \cup B \subseteq A_1 \cup B_1$ and since A_0 and B_0 are disjoint, we find $(A_1 \cup B_1) \setminus (A_0 \cup B_0) = (A_1 \setminus A_0) \cup (B_1 \setminus B_0)$. Thus

$$\mu((A_1 \cup B_1) \setminus (A_0 \cup B_0)) \leq \mu(A_1 \setminus A_0) + \mu(B_1 \setminus B_0) < \varepsilon$$

and so $A \cup B \in \mathscr{A}^*$ as desired.

(c) Let $A \in \mathscr{A}^*$. Then pick $A_0 \subseteq A \subseteq A_1$ such that $\mu(A_1 \setminus A_0) < \varepsilon$. But then we see $A_1^c \subseteq A^c \subseteq A_0^c$ and $\mu(A_0^c \setminus A_1^c) = \mu(A_1 \setminus A_0) < \varepsilon$ and so $A^c \in \mathscr{A}$ as desired. ■

Proposition B.5. $(X, \mathscr{A}^*, \mu^*)$ is a charge space.

Proof. Since $\varnothing \in \mathscr{A}$ and μ^* extends μ, we get $\mu^*(\varnothing) = \mu(\varnothing) = 0$. Now suppose $A, B \in \mathscr{A}^*$ are disjoint. Let $\varepsilon > 0$ be arbitrary. Pick $A_0 \subseteq A \subseteq A_1$ and $B_0 \subseteq B \subseteq B_1$ with $\mu(A_1 \setminus A_0) < \frac{\varepsilon}{2}$ and $\mu(B_1 \setminus B_0) < \frac{\varepsilon}{2}$. Without loss of generality, we can suppose that Without loss of generality, we can suppose A_1 and B_1 are disjoint by replacing A_1 with $A_1 \setminus B_1$. We then note that $\mu((A_1 \cup B_1) \setminus (A_0 \cup B_0)) = \mu(A_1 \cup B_1) - \mu(A_0 \cup B_0) = \mu(A_1) + \mu(B_1) - \mu(A_0) - \mu(B_0) < \varepsilon$ and so

$$\mu^*(A \cup B) \leq \mu(A_1 \cup B_1) < \mu(A_0) + \mu(B_0) + \varepsilon \leq \mu^*(A) + \mu^*(B) + \varepsilon$$

and similarly

$$\mu^*(A \cup B) \geq \mu(A_0 \cup B_0) > \mu(A_1) + \mu(B_1) - \varepsilon \geq \mu^*(A) + \mu^*(B) - \varepsilon.$$

This holds for all ε, so we get that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ as desired. ■

Proposition B.6. $\mathscr{A}^{**} = \mathscr{A}^*$

Proof. Let $A^{**} \in \mathscr{A}^{**}$. Pick $E^*, F^* \in \mathscr{A}^*$ such that $E^* \subseteq A^{**} \subseteq F^*$ and $\mu^*(F^* \setminus E^*) < \frac{\varepsilon}{3}$. In addition, pick $E \subseteq E^* \subseteq E'$ and $F \subseteq F^* \subseteq F'$ such that $\mu(E' \setminus E) < \frac{\varepsilon}{3}$ and $\mu(F' \setminus F) < \frac{\varepsilon}{3}$. In particular, since μ^* extends μ, we see that

$$\frac{\varepsilon}{3} > \mu(E' \setminus E) = \mu^*(E' \setminus E) \geq \mu^*(E^* \setminus E)$$

and similarly $\mu^*(F' \setminus F^*) < \frac{\varepsilon}{3}$. Thus we have $\mu^*(F' \setminus E) = \mu^*(F' \setminus F^*) + \mu^*(F^* \setminus E^*) + \mu^*(E^* \setminus E) < \varepsilon$. Since μ^* extends μ and $F' \setminus E \in \mathscr{A}$, we have $\mu(F' \setminus E) = \mu^*(F' \setminus E) < \varepsilon$ and $E \subseteq A^{**} \subseteq F'$. Then by Lemma B.3, we have $A^* * \in \mathscr{A}^*$ as desired. ■

B.2 The Space of Charges

The results here can be found in [9].

Definition B.7. Let X be a set and $\mathscr{A}$ be an algebra on X. We define the space of signed charges

$$\kappa_{\mathscr{A}}(X) = \{\mu : \mathscr{A} \to \mathbb{R} \mid \mu \text{ is a signed charge on } (X, \mathscr{A}) \text{ such that } \|\mu\| < \infty\}.$$

We equip $\kappa_{\mathscr{A}}(X)$ with the variation norm:

$$\left\|\mu\right\| = \sup\left\{\sum_{k=1}^{n}\left|\mu(A_k)\right| \;\middle|\; A_1, \ldots, A_n \text{ is an } \mathscr{A}\text{-partition of } X\right\}.$$

Theorem B.8. The space $\kappa(X, \mathscr{A})$ is complete with the variation norm.

References

[1] T. M. K. Davison. A generalization of regulated functions. *Amer. Math. Monthly*, 86(3):202–204, 1979.

[2] J. Dieudonne. *Foundations of Modern Analysis*. Pure and Applied Mathematics. Academic Press, 1969.

[3] Y. Guo, C. Huang, B. Pausader, and K. Widmayer. On the stabilizing effect of rotation in the 3d Euler equations. *Comm. Pure Appl. Math.*, 76(12):3553–3641, 2023.

[4] L. Khan. Mean value theorem in topological vector spaces. *Comptes Rendus Mathématiques de l'Académie des Sciences*, 19, 01 1997.

[5] J. Koganemaru and I. Tice. Traveling wave solutions to the inclined or periodic free boundary incompressible Navier-Stokes equations. *J. Funct. Anal.*, 285(7):Paper No. 110057, 75, 2023.

[6] G. Leoni and I. Tice. Traveling wave solutions to the free boundary incompressible Navier-Stokes equations. *Comm. Pure Appl. Math.*, 76(10):2474–2576, 2023.

[7] L. H. Loomis and S. Sternberg. *Advanced calculus*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1968.

[8] H. Nguyen and I. Tice. Traveling wave solutions to the one-phase muskat problem: existence and stability. *Arch. Rational Mech. Anal.*, to appear.

[9] K. P. B. Rao and M. B. Rao. *Theory of charges: A study of finitely additive measures*. Academic Press, 1983.

[10] N. Stevenson and I. Tice. Traveling wave solutions to the multilayer free boundary incompressible Navier-Stokes equations. *SIAM J. Math. Anal.*, 53(6):6370–6423, 2021.