Automatic Continuity of Group Homomorphisms

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21-800 Dynamics of Polish Groups

Contents

0 Introduction

In this report, we consider a question whose heart lies in a problem initially posed a long time ago by Cauchy: Is a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ continuous? More precisely, we'll be considering in various forms the question

For Polish groups G and H, when is a homomorphism $\varphi : G \to H$ continuous?

We will be following the exposition in Rosendal [1]. Indeed, it's not too hard to find an example where this question is nontrivial.

Example 0.1. $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic as $\mathbb{Q}$-vector spaces, and thus as groups - however, any such isomorphism (or bijection) must necessarily be discontinuous. To see this, suppose we had a continuous bijection $f : \mathbb{R} \to \mathbb{R}^2$. By the Baire category theorem, we could find some $n \in \mathbb{N}$ such that $f\big[[-n,n]\big]$ contained a closed ball $B \subseteq f\big[[-n,n]\big] \subseteq \mathbb{R}^2$. We then see that $f^{-1}[B]$ is closed and contained in $[-n,n]$, thus is compact, and so $f\restriction_{f^{-1}[B]}$ is a homeomorphism. In particular, $f^{-1}[B]$ is connected and so is a closed interval. Pick any $z \in B$ such that $f^{-1}[\{z\}]$ is not an endpoint of this interval. Then $f^{-1}[B] \setminus \{f^{-1}[\{z\}]\}$ is disconnected, but $B \setminus z$ remains connected, so we arrive at a contradiction.

Example 0.2. Let Z be a separable infinite dimensional Banach space.

1 Measurable Homomorphisms

To hone in on the problem then, we will restrict the classes of homomorphisms we will consider. Let us consider a few varying perspectives on measurability.

1.1 Baire Measurable Homomorphisms

We recall that for a Polish space X a set $E \subseteq X$ has the property of Baire if there is some Borel set B such that $E \triangle B$ is meager. This is the "category" perspective to enhance our collection of Borel sets.

Lucky for us, we've already solved this case through the work we've done! We recall Pettis' theorem as we proved in class.

Theorem 1.1 (Pettis)**.** Suppose G is a Polish group and $A, B \subseteq G$. Let $U(A)$ and $U(B)$ be the largest open sets in which A and B respectively are comeager. Then

$$U(A) \cdot U(B) \subseteq AB.$$

In particular, if A is nonmeager, then AA^{-1} is a neighborhood of the identity $e \in G$.

As a corollary of Pettis' theorem, we know that all Baire measurable homomorphisms are continuous. In particular, all Borel homomorphisms are continuous.

Corollary 1.2. Any Baire measurable homomorphism $\varphi : G \to H$ between Polish groups is continuous.

1.2 Haar Measurable Homomorphisms

We now work our way to understanding the "measure-theoretic" perspective on automatic continuity of homomorphisms. To begin, we define the analogue of the Baire measurable sets in the case of measure. Say G is a Polish group with Borel measure μ. We say that $E \subseteq X$ is μ-measurable if there is some Borel set B such that $E \triangle B$ is contained in μ-null set. In this case, we also define $\mu(E) = \mu(B)$. Indeed, this is essentially a direct translation from the category setting. When our Polish group carries a canonical measure, in particular when our Polish group is locally compact, we have a result in the affirmative for automatic continuity.

Theorem 1.3 (Weil)**.** Suppose G is a locally compact Polish group with left Haar measure μ. Then for any μ-measurable set $E \subseteq G$ with $\mu(E) > 0$, we have that

$$EE^{-1}$$

is a neighborhood of the identity $e \in G$.

The proof of this theorem is omitted, but is an elementary result in harmonic analysis after one has constructed a Haar measure. By a simple adaptation from the argument given in class, we get the corollary.

Corollary 1.4. Let G be a locally compact Polish group and H be a Polish group. Then any Haar measurable homomorphism $\varphi : G \to H$ is continuous.

1.3 Universally Measurable Homomorphisms

We now turn to the central result of this report. We say a set $E \subseteq G$ is universally measurable if for every Borel probability measure μ on G, we have that E is μ-measurable. This is a very annoying definition to work with in practice since the choice of our Borel "testing set" can vary with μ. The intuition is that universal measurability serves as a flag that all arguments about such sets should primarily be measure-theoretic and not involve in any meaningful way definability of Baire category.

We will endeavor to show that all universally measurable homomorphisms are in fact continuous through a series of lemmas, culminating in the key Lemma 1.9. We start with a useful technical result.

Lemma 1.5. Let G and H be Hausdorff topological groups, and $\varphi : G \to H$ a homomorphism. Define

$$N = \bigcap_{U \subseteq_e G} \overline{\varphi[U]} = \left\{ h \in H \;\middle|\; h = \lim_{\alpha} \varphi(g_\alpha) \text{ for some net} g_\alpha \to 1 \right\}.$$

Then N is a closed subgroup of H. Furthermore, if $\varphi[G]$ is dense in H, then N is the smallest closed normal subgroup of H so that the induced map $\tilde{\varphi} : G \to H/N$ has a closed graph.

Proof. It's obvious that N is closed. To see that N is a subgroup of H, let $U \subseteq_e G$ and pick $V \subseteq_e G$ such that $VV^{-1} \subseteq U$. We then see

$$\overline{\pi[V]} \cdot \overline{\pi[V]}^{-1} \subseteq \overline{\pi[V]} \cdot \overline{\pi[V^{-1}]} \subseteq \overline{\pi[VV^{-1}]} \subseteq \overline{\pi[U]}.$$

In particular, we conclude that $NN^{-1} \subseteq N$ and so $N \leq H$.

To check that N is normal, we wish to show that $hNh^{-1} \subseteq N$ for all $h \in H$. Since N is closed and $\pi[G]$ is dense, it suffices to show that $\varphi(g)N\varphi(g)^{-1} \subseteq N$ - indeed, by continuity of multiplication this readily implies $hNh^{-1} \subseteq \overline{N} = N$ for all $h \in H$. Towards that end, fix $n \in N$ and $g \in G$ and let $U \subseteq_e G$. Then $g^{-1}Ug \subseteq_e G$ as well, so we have

$$n \in \overline{\varphi[g^{-1}Ug]} = \overline{\varphi(g)^{-1}\varphi[U]\varphi(g)} = \varphi(g)^{-1}\overline{\varphi[U]}\varphi(g).$$

In particular, we have that $\varphi(g)n\varphi(g)^{-1} \in \overline{\pi[U]}$ for all $n \in N$ and $U \subseteq_e G$, which gives that $\varphi(g)N\varphi(g)^{-1} \subseteq N$ as desired.

Now let H/N be the quotient group equipped with the quotient topology (which is Hausdorff as N is closed). Let $\tilde{\varphi} : G \to H/N$ be the induced map. We will endeavor to show that the graph $\mathcal{G}\tilde{\varphi} \subseteq G \times H/N$ is closed. Note that because $\tilde{\varphi}$ is a homomorphism, we have that $\mathcal{G}\tilde{\varphi}$ is a subgroup of $G \times H/N$, and hence so is its closure $\overline{\mathcal{G}\tilde{\varphi}}$. In particular, if $(g, hN) \in \overline{\mathcal{G}\tilde{\varphi}} \setminus \mathcal{G}\tilde{\varphi}$, we have that $(g^{-1}, \varphi(g)^{-1}) \in \mathcal{G}\tilde{\varphi}$ and so $(g^{-1}, \varphi(g)^{-1})(g, hN) = (e, \varphi(g)^{-1}hN) \in \overline{\mathcal{G}\tilde{\varphi}} \setminus \mathcal{G}\tilde{\varphi}$. Thus to show that $\overline{\mathcal{G}\tilde{\varphi}} \setminus \mathcal{G}\tilde{\varphi}$ is empty, it suffices to show that if $g \notin N$ then $(e, gN) \notin \mathcal{G}\tilde{\varphi}$.

Towards that end, fix $g \notin N$. Pick some $U \subseteq_e G$ such that $g \notin \overline{\varphi[U]}$, and by Hausdorffness some

$V \subseteq_g G$ such that $V \cap \overline{\varphi[U]} = \varnothing$. Since U is open, we have that $\varphi[U]N \subseteq \overline{\varphi[U]}$, so we can conclude that $VN \cap \varphi[U]N = \varnothing$, which readily implies that $U \cap VN/N$ is a neighborhood of (e, gN) disjoint from $\mathcal{G}\tilde{\varphi}$, which gives that $(e, gN) \notin \overline{\mathcal{G}\tilde{\varphi}}$ as desired.

Finally, to see that N is the smallest such subgroup, fix some $K \trianglelefteq G$ such that the induced map $\hat{\varphi} : G \to H/K$ has closed graph. Pick $n \in N$ and some net $(g_\alpha)_\alpha$ such that $g_\alpha \to e$ and $\varphi(g_\alpha) \to n$. Thus we get $\varphi(g_\alpha)K \to nK$, which implies that (e, nK) is in $\overline{\mathcal{G}\hat{\varphi}} = \mathcal{G}\hat{\varphi}$ - in other words $n \in K$, and we're done! ■

In the interest of brevity, we skip the proofs of the next two lemmas as they are easier to establish.

Lemma 1.6. Let G and H be Polish groups and $\varphi : G \to H$ a homomorphism. Suppose that $X \subseteq G$ is comeager and $U \subseteq G$ is open. Then $\overline{\varphi[U]} = \overline{\varphi[U \cap X]}$.

Lemma 1.7. Let N be a Hausdorff topological group with the property that for $U \subseteq_e N$, there is a finite set F such that $N = \bigcup_{f \in F} fUf^{-1}$. Then $N = \{e\}$.

Before we arrive at Lemma 1.9, we recall a result of Christensen [2] which will be critical in establishing automatic continuity of universally measurable homomorphisms. Although we don't use the result, it provides motivation for our key lemma.

Theorem 1.8 (Christensen). Suppose G is Polish and $\{A_k\}_{k \in \mathbb{N}}$ is a collection of universally measurable subsets of G such that $G = \bigcup_{k \in \mathbb{N}} A_k$. Let $U \subseteq_e G$. Then there exists $u_1, \ldots, u_m \in U$ and some $k \in \mathbb{N}$ such that

$$\bigcup_{j=1}^{m} u_j A_k A_k^{-1} u_j^{-1}$$

is a neighborhood of e in G.

Christensen's result without too much effort implies that universally measurable homomorphisms from a Polish group G to a TSI group H are automatically continuous. We note this is quite similar to the Steinhaus properties we were studying previously in class.

Lemma 1.9. Let G and H be Polish groups, and let $\varphi : G \to H$ be a homomorphism. Suppose that for all $U \subseteq_e G$ and $V \subseteq_e H$ that there is a finite set $M \subseteq U$ such that

$$\bigcup_{g \in M} g \cdot \varphi^{-1}[V] \cdot g^{-1}$$

is comeager in an open neighborhood of $e \in G$. Then φ is continuous!

Proof. We may assume without loss of generality that $\varphi[G]$ is dense in H, as we can always consider the map $\varphi : G \to \overline{\varphi[G]}$ and $\overline{\varphi[G]}$ is also a Polish group. Then let N and $\tilde{\varphi}$ be as in Lemma 1.5. We then know that $\tilde{\varphi} : G \to H/N$ has closed graph, and since both G and H/N are Polish, we conclude that $\tilde{\varphi}$ is continuous. Thus to see that φ is continuous, all we need to do is show that N is trivial, or equivalently by Lemma 1.7 that we can find a finite set $F \subseteq N$ with $N \subseteq \bigcup_{f \in F} fVf^{-1}$ for every $V \subseteq_e H$.

Towards that end, let $V \subseteq_e H$ and pick $W \subseteq_e H$ such that $\overline{WWW^{-1}} \subseteq V$. Since $\tilde{\varphi}$ is continuous, we see that $\varphi^{-1}(NW) \subseteq_e G$. By assumption, we can find finite $M \subseteq \varphi^{-1}(NW)$ such that

$$B := \bigcup_{g \in M} g\varphi^{-1}(W)g^{-1}$$

us comeager in some $U \subseteq_e G$. In particular, by Lemma 1.6, we have $N \subseteq \overline{\varphi[U]} \subseteq \overline{\varphi[B]}$. Letting $F \subseteq N$ be a finite set with $\varphi[M] \subseteq FW$, we have

$$\varphi[B] \subseteq \bigcup_{g \in M} \varphi(g)W\varphi(g)^{-1} \subseteq \bigcup_{f \in F} fWWW^{-1}f^{-1}$$

which then implies as desired

$$N \subseteq \overline{\varphi[B]} \subseteq \bigcup_{f \in F} f\overline{WWW^{-1}}f^{-1} \subseteq \bigcup_{f \in F} fVf^{-1}. \qquad \blacksquare$$

Finally, we show our desired conclusion.

Theorem 1.10. Let G and H be Polish groups and suppose $\varphi : G \to H$ is a universally measurably homomorphism. Then φ is continuous.

Proof. To prove this, we just need to verify the hypotheses of Lemma 1.9. To do this, we'll leverage Theorem 1.8. Suppose $V \subseteq_e H$ and pick $W \subseteq_e H$ such that $WW^{-1} \subseteq V$. By separability of H, pick

$\{h_n\}_n \subseteq H$ such that $\bigcup_n h_n W = H$. Then $A_n := \varphi^{-1}[h_n W]$ is universally measurable, and we see that $G = \varphi^{-1}[H] = \bigcup_n A_n$. By Theorem 1.8, this means for each $U \subseteq_e G$ we can find $u_1, \dots, y_m \in U$ and some $n \in \mathbb{N}$ such that

$$\bigcup_{j=1}^{m} u_j A_n A_n^{-1} u_j^{-1}$$

is a neighborhood of $e \in G$. We note that $A_n A_n^{-1} = \varphi^{-1}[h_n W]\left(\varphi^{-1}[h_n W]\right)^{-1} \subseteq \varphi^{-1}[h_n W(h_n W)^{-1}] = \varphi^{-1}[WW^{-1}] \subseteq \varphi^{-1}[V]$. Thus

$$\bigcup_{j=1}^{m} u_j \varphi^{-1}[V] u_j^{-1}$$

is a neighborhood of the identity, and so by Lemma 1.9 proves the theorem. ■

Remark 1.11. Both Lemma 1.9 and Theorem 1.10 can be shown for H just separable rather than Polish, but the modifications are simply technical details which we have omitted to highlight the heart of the argument.

2 References

[1] C. Rosendal, 'Continuity of universally measurable homomorphisms', *Forum of Mathematics*. Pi (2019), Vol. 7, e5

[2] J. P. R. Christensen, 'On sets of Haar measure zero in abelian Polish groups', *Israel J. Math.* 13 (1972), 255–260