Singular Integrals

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The main motivating example for a singular integral is given by the *Hilbert transform*, H, defined as

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\, \mathrm{d}y.$$

We showed that $1/x \in \mathscr{D}'$ when taken in this principal value sense. With a bit more work, we can show $1/x \in \mathscr{S}'$, so the Hilbert transform is a convolution operator with this distribution.

This can be considered as an endpoint case in the one-dimensional Riesz potentials. Although a priori the integral of $1/x$ is too singular to make sense of, we will show that in fact if $f \in L^p(\mathbb{R})$ for $1 < p < \infty$, then Tf exists almost everywhere and is in $L^p(\mathbb{R})$, with T defining a bounded operator on $L^p(\mathbb{R})$.

T is not bounded on L^1, but is of weak-type $(1,1)$.

A part of the reason we can make sense of the Hilbert transform and why it serves as a good starting point for the study of singular integrals is due to certain symmetries it has. Harmonic analysis is, after all, a study of how much the symmetries of our space can buy us analytically. We highlight these here:

- The Hilbert transform commutes with translations, the action of $(\mathbb{R}, +)$
- The Hilbert transform commutes with positive dilations, the action of $((0,\infty), \times)$
- The Hilbert transform commutes with reflections, the action of $(\{1,-1\}, \times)$

All of these follow readily with simple computations. In fact, as an operator on L^2 (where we have the full power of Plancherel), these properties characterize the Hilbert transform up to a constant. Working with all or some of these symmetries, we can get a rich theory of integrals we shouldn't be able to make sense of. Unfortunately, we won't have time for that in these notes, but it is worth noting that the different forms in which singular integrals appear touch on a lot of areas of analysis.

Translation Invariant Transformations

Our study in this report will largely be devoted to special translation invariant transformations on L^p spaces, i.e. transformations T that satisfy $T\tau_z = \tau_z T$ for all $z \in \mathbb{R}^n$. We recall a few theorems, stating that any such transformation is given as a convolution.

Theorem 1. *An operator $T : L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is linear, continuous, and translation invariant if and only if there is a measure of bounded variation such that $Tf = \mu * f$ for all $f \in L^1$.*

Theorem 2. *An operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is linear, continuous, and translation invariant if and only if there is a bounded function $m : \mathbb{R}^n \to \mathbb{R}$ such that $\widehat{Tf} = m\hat{f}$ for all $f \in L^2$.*

In this case, m is called a *multiplier*. Often when analyzing transformations of this form, it is simpler to work with the multiplier rather than the convolution with the original transformation. To finish, we have a more general theorem.

Theorem 3. *Suppose that $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is linear, continuous, and translation invariant. Then there is a unique $K \in \mathscr{S}'$ such that $T\varphi = \varphi * K$ for all $\varphi \in \mathscr{S}$.*

Thus if we wish to understand translation invariant operators, we really need to understand convolutions. The theorems above show that translation invariant operators on L^1 and L^2 are well understood. Translation invariant operators on generic L^p are a little more complicated. The next theorem starts our foray into this analysis.

The Essence of the Argument

As a fair warning, because we'll be doing a lot of bounding, I'm going to quite liberally use C to mean a generic constant that changes from line to line.

Theorem 4. *Let $K \in L^2(\mathbb{R}^n)$. Suppose that $\widehat{K} \in L^\infty$ with $\left\|\widehat{K}\right\|_{L^\infty} \leq B$. and that there is some $D > 0$ such that for each $y \neq 0$, we have*

$$\int_{|x|>|2y|} |K(x-y) - K(x)| \, \mathrm{d}x \leq D$$

For $f \in L^1 \cap L^p$, define Tf by

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y) \, \mathrm{d}y.$$

Then if $1 < p < \infty$, there exists a constant $A = A(p, n, B, D)$ such that $\|Tf\|_{L^p} \leq A\|f\|_{L^p}$. In particular, T extends to a bounded operator on L^p. We can note that the constant A does not depend on the L^2 norm of K.

Proof. Our goal will be to use the Marcinkiewicz interpolation theorem as well as a duality argument. With this in mind, we split our proof up into steps.

Step 1 - T is of strong type $(2,2)$: For $f \in L^1 \cap L^2$, we see that $\widehat{Tf}(x) = \widehat{K}(x)\hat{f}(x)$ since T is just a convolution. This readily implies that $\|Tf\|_{L^2} = \left\|\widehat{Tf}\right\|_{L^2} \leq B\|f\|_{L^2}$ since $\widehat{K}(x) \leq B$ almost everywhere, and so T extends to a bounded operator on L^2.

Step 2 - T is of weak type $(1, 1)$: The last step was simple, but this step will require a bit more work. Unpacking, our goal in this step is to find a constant C such that for each $\alpha > 0$ we have

$$\mu\{|Tf| > \alpha\} \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(x)| \, \mathrm{d}x. \tag{1}$$

Throughout this step, we use C to be a constant that possibly depends on n, B, and D, but it can change from line to line. To accomplish our task, we're going to need a lemma important enough to have a name and crucial to the development of singular integrals.

The Argentine mathematician Alberto Calderón was initially a student and later a long time collaborator of the Polish analyst Antoni Zygmund They are largely credited with the development of the theory of singular integrals, particularly in their seminal work "On the existence of singular integrals". Elias Stein (also a student of Zygmund) wrote in an essay in 1998: "There is probably no paper in the last fifty years which had such widespread influence in analysis". The lemma proved here first appeared in that paper.

Lemma 5 (Calderón-Zygmund Lemma)**.** *Let $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$ almost everywhere and let $\alpha > 0$. Then there exists an opens set $\Omega \subseteq \mathbb{R}^n$ such that*

(i) There are countably many disjoint open cubes $\mathcal{Q} \subseteq \mathcal{P}(\mathbb{R}^n)$ with $\Omega = \bigcup \mathcal{Q}$ and for each $Q \in \mathcal{Q}$, we have

$$\alpha < \frac{1}{\mu(Q)} \int_Q f(x) \, \mathrm{d}x \leq 2^n \alpha,$$

(ii) $f \leq \alpha$ almost everywhere on Ω^c, and

(iii) $\mu(\Omega) \leq \frac{1}{\alpha} \|f\|_{L^1}$.

Proof. Decompose $\mathbb{R}^n$ (excepting a null set) into a mesh of congruent open cubes such that each cube has measure at least $\frac{1}{\alpha}\|f\|_{L^1}$. In particular, this implies that for each cube Q' in this mesh we have $\int_{Q'} f \leq \int_{\mathbb{R}^n} f \leq \alpha\mu(Q')$. We now proceed with an algorithm for dividing these cubes further.

Fix a cube Q' in this mesh. By halving each of the sides, we can divide Q' into 2^n congruent smaller open cubes. Let Q be one of these subdivided cubes. If $\frac{1}{\mu(Q)} \int_Q f > \alpha$, we let $Q \in \mathcal{Q}$ as in (ii). In particular, we get

$$\alpha < \frac{1}{\mu(Q)} \int_Q f = \frac{2^n}{\mu(Q')} \int_Q f \leq \frac{2^n}{\mu(Q')} \int_{Q'} f \leq 2^n \alpha.$$

On the other hand, if $\frac{1}{\mu(Q)} \int_Q f \leq \alpha$, repeat the splitting and keep going.

We then define $\Omega = \bigcup \mathcal{Q}$. Now (ii) is done by the (generalized) Lebesgue differentiation theorem. Indeed, for almost every $x \in \Omega^c$, we obtain that

$$f(x) = \lim_{\substack{|Q| \to 0 \\ x \in Q}} \frac{1}{\mu(Q)} \int_Q f(y) \, \mathrm{d}y \leq \alpha$$

where the second inequality followed by the definition of Ω^c. Finally, (iii) follows by noting $\mu(\Omega) = \sum_{Q \in \mathcal{Q}} \mu(Q) < \sum_{Q \in \mathcal{Q}} \frac{1}{\alpha} \int_Q f = \frac{1}{\alpha}\|f\|_{L^1}$ and so we're done. ■

The Calderón-Zygmund lemma is equivalent to the inequality $\|Mf\|_{L^{1,\infty}} \leq C\|f\|_{L^1}$ for the maximal function - they can be used to prove each other without e.g. a Vitali covering argument.

Now we will see the power of Calderón and Zygmund's argument. Towards proving (1), fix $\alpha > 0$. Apply the Calderón-Zygmund lemma to $|f|$ and obtain Ω and $\mathcal{Q}$ as in the theorem. Define the "good" function $g : \mathbb{R}^n \to \mathbb{R}$ by setting

$$g(x) = \begin{cases} f(x) & x \in \Omega^c \\ \frac{1}{\mu(Q)} \int_Q f & x \in Q, Q \in \mathcal{Q} \end{cases}.$$

This defines g almost everywhere. We then define the "bad" function b by setting $b = f - g$. In particular, we see that $b(x) = 0$ for $x \in \Omega^c$ and for each $Q \in \mathcal{Q}$ we have $\int_Q b = 0$. We have $Tf = Tg + Tb$, so $\mu\{|Tf| > \alpha\} \leq \mu\{|Tg| > \alpha/2\} + \mu\{|Tb| > \alpha/2\}$, so we will prove the analogue of the bound (1) for each in turn.

To get the bound for Tg, we will prove that in fact $g \in L^2$. We first note that for each $x \in \Omega$, then $x \in Q$ for some $Q \in \mathcal{Q}$ and we have that $g(x) = \frac{1}{\mu(Q)} \int_Q f \leq \frac{1}{\mu(Q)} \int_Q |f| \leq C\alpha$ by construction in the Calderón-Zygmund lemma, and $|g| = |f| \leq \alpha$ on Ω^c. Using this, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |g|^2 &= \int_{\Omega^c} |g|^2 + \int_\Omega |g|^2 \\ &\leq \alpha \int_{\Omega^c} |f| + C^2\alpha^2 \mu(\Omega) \leq C\alpha \|f\|_{L^1} < \infty. \end{aligned}$$

This and our work in Step 1 (combining with the fact that $|||g|||_{L^{2,\infty}} \leq \|g\|_{L^2}$) tells us that

$$\mu\{|Tg| > \alpha\} \leq \frac{C}{\alpha^2} \|g\|_{L^2}^2 \leq \frac{C}{\alpha} \|f\|_{L^1}$$

and so we have the desired bound.

To finish this step, we prove the bound for Tb. As it will turn out, b is not so bad. For $Q \in \mathcal{Q}$, we define c_Q to be the center of the cube and s_Q to be the side length of the cube. We let $b_Q = b\chi_Q$, so $b = \sum_{Q \in \mathcal{Q}} b_Q$ and $Tb = \sum_{Q \in \mathcal{Q}} Tb_Q$. Additionally, we define Q^* to be the cube with center c_Q and side length $2\sqrt{n}s_Q$. From this, we set $\Omega^* = \bigcup_{Q \in \mathcal{Q}} Q^*$. Then $\Omega \subseteq \Omega^*$ and

$$\mu(\Omega^*) \leq \sum_{Q \in \mathcal{Q}} \mu\left(Q^*\right) = \left(2\sqrt{n}\right)^n \sum_{Q \in \mathcal{Q}} \mu\left(Q\right) = \left(2\sqrt{n}\right)^n \mu(\Omega).$$

Furthermore, if $x \notin Q^*$ and $y \in Q$, then

$$\left|x - c_Q\right| \geq 2\sqrt{n}s_Q \geq 2\left|y - c_Q\right|. \tag{$*$}$$

With the pieces in play, we begin our bounding.

We first note since b is mean zero on any $Q \in \mathcal{Q}$, we have

$$Tb_Q(x) = \int_Q K(x-y)b(y)\,\mathrm{d}y = \int_Q \left(K(x-y) - K(x-c_Q)\right) b(y)\,\mathrm{d}y$$

Integrating Tb, we then find

$$\begin{aligned}
\int_{(\Omega^*)^c} |Tb(x)| \, dx &\leq \sum_{Q \in \mathcal{Q}} \int_{(\Omega^*)^c} \int_Q \left| K(x-y) - K(x-c_Q) \right| |b(y)| \, dy \, dx \\
&\leq \sum_{Q \in \mathcal{Q}} \int_{(Q^*)^c} \int_Q \left| K(x-y) - K(x-c_Q) \right| |b(y)| \, dy \, dx \\
&= \sum_{Q \in \mathcal{Q}} \int_Q |b(y)| \int_{(Q^*)^c} \left| K(x-y) - K(x-c_Q) \right| \, dx \, dy
\end{aligned}$$

We now estimate the inner integral. Fix $y \in Q$ and $x \in (Q^*)^c$ and let $y' = y - c_Q$ and $x' = x - c_Q$. Then we see that $|x'| \geq 2|y'|$ by the bound $(*)$. In particular, by changing variables accordingly we get for fixed $y \in Q$

$$\int_{(Q^*)^c} \left| K(x-y) - K(x-c_Q) \right| \, dx \leq \int_{|x'| \geq 2|y'|} \left| K(x'-y') - K(x') \right| \, dx' \leq D.$$

In addition, we also estimate

$$\begin{aligned}
\int_\Omega |b(y)| \, dy &\leq \int_\Omega |f(y)| \, dy + \int_{\mathbb{R}^n} |g(y)| \, dy \\
&\leq \int_\Omega |f(y)| \, dy + C\alpha\mu(\Omega) \leq C\|f\|_{L^1}
\end{aligned}$$

With these facts, our bound for the integral of Tb becomes

$$\begin{aligned}
\int_{(\Omega^*)^c} |Tb(x)| \, dx &\leq \sum_{Q \in \mathcal{Q}} \int_Q |b(y)| \int_{(Q^*)^c} \left| K(x-y) - K(x-c_Q) \right| \, dx \, dy \\
&\leq B \int_\Omega |b(y)| \, dy \leq C\|f\|_{L^1}
\end{aligned}$$

And now we can see the end. Indeed, now Chebyshev's inequality implies

$$\mu\left\{ x \in (\Omega^*)^c \;\middle|\; \alpha/2 < |Tb(x)| \right\} \leq \frac{C}{\alpha} \|f\|_{L^1}$$

On the other hand, we have $\mu(\Omega^*) \leq C\mu(\Omega) \leq \frac{C}{\alpha}\|f\|_{L^1}$. Thus

$$\mu\{x \in \mathbb{R}^n \mid \alpha/2 < |Tb(x)|\} \leq \mu(\Omega^*) + \mu\left\{ x \in (\Omega^*)^c \;\middle|\; \alpha/2 < |Tb(x)| \right\} \leq \frac{C}{\alpha} \|f\|_{L^1}.$$

Step 3 - The inequalities for $1 < p \leq 2$: We showed that T is of weak type $(1,1)$ and strong type $(2,2)$, so the Marcinkiewicz interpolation theorem graces us with the fact that T is a bounded operator on L^p for all $1 < p \leq 2$, i.e. $\|Tf\|_{L^p} \leq A\|f\|_{L^p}$ with the constant A depending only on p, n, B, and D.

Step 4 - The inequalities for $2 < p < \infty$: Fix such a p. Here we will leverage duality, so let q be such that $\frac{1}{q} + \frac{1}{p} = 1$. We recall that for any measurable $\psi : \mathbb{R}^n \to \mathbb{R}$, if we have

$$\sup_{\substack{\varphi \in L^q \\ \|\varphi\|_{L^q} \leq 1}} \int_{\mathbb{R}^n} |\psi|\varphi = A < \infty,$$

then in fact $\psi \in L^p$ and $\|\psi\|_{L^p} = A$. By density, the supremum over $\varphi \in L^q$ can be weakened to $\varphi \in C_c^\infty(\mathbb{R}^n)$. With this in mind, fix $f \in L^1 \cap L^p$ and $\varphi \in C_c^\infty$ with $\|\varphi\|_{L^q} \leq 1$. By Young's inequality, we see that $K * f \in L^2$, and so $\varphi(K * f)$ is in L^1. In particular, we have that the integral

$$\int_{\mathbb{R}^n} K * f(x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)f(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x$$

converges. Fubini then lets us say its value is

$$I = \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n} K(x-y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^n} f(y)\rho K * \varphi(y)\,\mathrm{d}y.$$

However, since we have proved the boundedness for L^q in Step 3, we can apply that result to T defined with ρK instead of K and find some constant A with $\|\rho K * \varphi\|_{L^q} \leq A$. Hölder's inequality gives us that $\left|\int_{\mathbb{R}^n} Tf(x)\varphi(x)\,\mathrm{d}x\right| = |I| \leq A\|f\|_{L^p}$, and taking the supremum over all such φ gives us that $\|Tf\|_{L^p} \leq A\|f\|_{L^p}$ as desired. ■

We note that the condition that $\int_{|x|>|2y|}|K(x-y)-K(x)|\,\mathrm{d}x$ is bounded is implied by more tangible things. For example, if $K \in C^1(\mathbb{R}^n \setminus \{0\})$ and $|\nabla K(x)| \leq \frac{C}{|x|^{n+1}}$ for some constant C, then we see for $|x| > 2|y|$ that $\frac{1}{2}|x| \leq |x-ty|$ for each $t \in [0,1]$, so we have by the mean value theorem

$$|K(x-y)-K(x)| \leq \sup_{t\in[0,1]}|\nabla K(x-ty)||y| \leq \sup_{t\in[0,1]}\frac{C|y|}{|x-ty|^{n+1}} = C\frac{|y|}{|x|^{n+1}}.$$

Integrating, we then find

$$\int_{|x|>|2y|}|K(x-y)-K(x)|\,\mathrm{d}x \leq C|y|\int_{|x|>|2y|}\frac{1}{|x|^{n+1}} \leq C$$

as desired. Additionally, although we've definitely proved some nontrivial bounds, the analysis might feel a little unsatisfactory - we are using that $K \in L^2$ instead of leveraging some symmetries of K to prove what we want. Our current theorem does not yet handle the case of the Hilbert transform, which is a "principal value" singular integral, dealing with the cancellation of positive and negative values. For this, we look to our next result.

Extending the Theorem

Theorem 6. *Suppose there exists $B, D > 0$ such that the kernel $K : \mathbb{R}^n \to \mathbb{R}$ satisfies $|K(x)| \leq B|x|^{-n}$ for almost every x,*

$$\int_{|x|>|2y|}|K(x-y)-K(x)|\,\mathrm{d}x \leq B,$$

and the cancellation property

$$\int_{R_1<|x|<R_2} K(x)\,\mathrm{d}x = 0$$

for each $0 < R_1 < R_2 < \infty$. For $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$, define the function $T_\varepsilon f$ by

$$T_\varepsilon f(x) = \int_{|y|>\varepsilon} f(x-y)K(y)\ \mathrm{d}y.$$

Then there is some constant $A > 0$ independent of f and ε such that

$$\|T_\varepsilon f\|_{L^p} \leq A\|f\|_{L^p}.$$

Additionally, there is some function $Tf \in L^p$ such that $\lim_{\varepsilon\to 0} T_\varepsilon f = Tf$ in L^p. Defining T in this way, T also satisfies the above bound.

Proof. Define $K_\varepsilon = K\chi_{B(0,\varepsilon)^c}$. Because $|K(x)| \leq B|x|^{-n}$, we see for each $p > 1$ that $|K(x)|^p \leq B^p|x|^{-np}$, and $K_\varepsilon \in L^p$ for all such p. In particular, with an eye towards the previous theorem, we have $K_\varepsilon \in L^2$. We now look to estimate $\widehat{K_\varepsilon}$ so we can apply Theorem 4. We delegate this to the following lemma.

Lemma 7. *There is some constant $C > 0$ depending only on n such that for each $\varepsilon > 0$ we have*

$$\left\|\widehat{K_\varepsilon}\right\|_{L^\infty} \leq CB.$$

Proof. We first prove the estimate for the special case $\varepsilon = 1$. We will then leverage a scaling trick to get the general estimate. Throughout the proof, we use C to denote a constant that depends only on n.

It's clear that $\int_{R_1<|x|<R_2} K_1(x)\ \mathrm{d}x = 0$ for all $0 < R_1 < R_2 < \infty$ and that $|K_1(x)| \leq B|x|^{-n}$. We also claim that we have

$$\int_{|x|>|2y|} |K_1(x-y) - K_1(x)|\ \mathrm{d}x \leq CB.$$

Indeed, we can note that we can split

$$\begin{aligned}
\int_{|x|>|2y|} |K_1(x-y) - K_1(x)|\ \mathrm{d}x &= \int_{\substack{|x|>|2y| \\ |x|\geq 1 \\ |x-y|\geq 1}} |K(x-y) - K(x)|\ \mathrm{d}x \\
&\quad + \int_{\substack{|x|>|2y| \\ |x|<1 \\ |x-y|\geq 1}} |K(x-y)|\ \mathrm{d}x + \int_{\substack{|x|>|2y| \\ |x|\geq 1 \\ |x-y|<1}} |K(x)|\ \mathrm{d}x \\
&\leq B + \int_{1\leq|x-y|\leq 2} \frac{B}{|x-y|^n}\ \mathrm{d}x + \int_{1\leq|x|\leq 2} \frac{B}{|x|^n}\ \mathrm{d}x \\
&\leq CB
\end{aligned}$$

as desired. Now, towards the proof of the lemma, we estimate

$$\begin{aligned}
\widehat{K_1}(y) &= \lim_{R\to\infty} \int_{|x|\leq R} e^{2\pi i x\cdot y} K_1(x)\ \mathrm{d}x \\
&= \underbrace{\int_{|x|\leq \frac{1}{|y|}} e^{2\pi i x\cdot y} K_1(x)\ \mathrm{d}x}_{I} + \underbrace{\lim_{R\to\infty} \int_{\frac{1}{|y|}\leq|x|\leq R} e^{2\pi i x\cdot y} K_1(x)\ \mathrm{d}x}_{II}
\end{aligned}$$

By the cancellation condition, we see that $I = \int_{|x|\le \frac{1}{|y|}} \left(e^{2\pi i x\cdot y} - 1\right) K_1(x)\, \mathrm{d}x$, so

$$|I| \le C|y| \int_{|x|\le \frac{1}{|y|}} |x||K_1(x)|\, \mathrm{d}x \le CB|y| \int_{|x|\le \frac{1}{|y|}} \frac{1}{|x|^n}\, \mathrm{d}x = CB.$$

For the second estimate, let $z = \frac{1}{2}\frac{y}{|y|^2}$, so we have $e^{2\pi i y\cdot z} = -1$ and $|z| = \frac{1}{2|y|}$. Then we see $\int_{\mathbb{R}^n} K_1(x) e^{2\pi i x\cdot y}\, \mathrm{d}x = \int_{\mathbb{R}^n} K_1(x-z) e^{2\pi i (x-z)\cdot y}\, \mathrm{d}x = -\int_{\mathbb{R}^n} K_1(x-z) e^{2\pi i x\cdot y}\, \mathrm{d}x$, so

$$\begin{aligned} II &= \lim_{R\to\infty} \int_{\frac{1}{|y|}\le |x|\le R} K_1(x) e^{2\pi i x\cdot y}\, \mathrm{d}x \\ &= \lim_{R\to\infty} \frac{1}{2} \int_{\frac{1}{|y|}\le |x|\le R} \left(K_1(x) - K_1(x-z)\right) e^{2\pi i x\cdot y}\, \mathrm{d}x - \frac{1}{2} \int_{\substack{\frac{1}{|y|} < |x+z| \\ |x| < \frac{1}{|y|}}} K_1(x) e^{2\pi i x\cdot y}\, \mathrm{d}x \end{aligned}$$

The last integral is taken over a subset of $\left\{ x \in \mathbb{R}^n \;\middle|\; \frac{1}{2}|y| \le |x| \le \frac{1}{|y|} \right\}$ and so is bounded by the fact that $|K_1(x)| \le B|x|^{-n}$. For the first integral, we bound above for all R by $\int_{\frac{1}{|y|}\le |x|} |K_1(x) - K_1(x-z)|\, \mathrm{d}x = \int_{2|z|\le |x|} |K_1(x) - K_1(x-z)|\, \mathrm{d}x \le CB$. This finishes the lemma for the case $\varepsilon = 1$.

For generic ε, define K' by $K'(x) = \varepsilon^n K(\varepsilon x) = \varepsilon^n \delta_\varepsilon K(x)$. Some simple change of variables shows that K' satisfies the condition of our theorem with the same bounds, so we define K_1' in analogy with K_1 and note that by our previous work we have $\left|\widehat{K_1'(\varepsilon y)}\right| \le CB$, but then we have

$$CB \ge \left|\widehat{K_1'(\varepsilon y)}\right| = \left|\varepsilon^{-n} \widehat{\delta_{\varepsilon^{-1}} K}(y)\right| = \left|\widehat{K_\varepsilon}(y)\right|$$

and we're done. ■

We note that in our lemma, we proved that K_1 satisfies the bounds in the theorem (with some extra dimensional constant). With an analogous dilation argument, we can also see that K_ε satisfies the bounds in the theorem. Thus using Theorem 4, we find some constant $A > 0$ independent of f and ε such that $\|T_\varepsilon f\|_{L^p} \le A\|f\|_{L^p}$.

Finally, we show that T is well defined in the limit. For $\varphi \in C_c^\infty$, we see that

$$T_\varepsilon \varphi(x) = \int_{|y|>1} K(y)\varphi(x-y)\, \mathrm{d}y + \int_{\varepsilon\le |y|\le 1} K(y)\left(\varphi(x-y) - \varphi(x)\right)\, \mathrm{d}y$$

by the cancellation condition for K. We recall that $K\varepsilon \in L^p$ and clearly $\varphi \in L^1$, so the first integral is just $K\chi_{B(0,1)^c} * \varphi(x)$ which is an L^p function by Young's inequality. The second integral, taken as a function of x, is supported in a fixed compact set and converges uniformly in x as $\varepsilon \to 0$ because $\varphi \in C_c^\infty$ and $|\varphi(x-y) - \varphi(x)| \le C|y|$. Taken together, we get that $T_\varepsilon \varphi$ converges in L^p as $\varepsilon \to 0$. Density of C_c^∞ then grants us this continuity for all of L^p, which gives the desired result. ■

And now we have what we need to handle the Hilbert transform! Indeed $\frac{1}{\pi x}$ obviously satisfies the conditions of the theorem when $n = 1$, so the Hilbert transform is bounded in L^p for all $1 < p < \infty$. This final version of our theorem handles many cases of interest, but there is much more to study when it comes to singular integrals. We haven't yet leveraged the other symmetries pointed out in the Hilbert transform. Before signing off, we briefly discuss this.

Dilation Invariant Operators

For an operator T to commute with dilations in addition to translations, we would need that $T\delta_\lambda = \delta_\lambda T$ or each $\lambda > 0$, or $T = \delta_{\lambda^{-1}} T \delta_\lambda$. With that in mind, suppose that T corresponds to convolution with the kernel K. As we discussed above, then $\delta_{\lambda^{-1}} T \delta_\lambda$ corresponds to convolution with the kernel $\lambda^{-n}\delta_{\lambda^{-1}} K$. Put another way, we have $K(\lambda x) = \lambda^{-n} K(x)$ for each $x \in \mathbb{R}^n$, i.e. that K is homogenous of degree $-n$. Put another way, we have

$$K(x) = \frac{\Omega(x)}{|x|^n}$$

for $\Omega : \mathbb{R}^n \to \mathbb{R}$ that is homogenous of degree 0. The cancellation condition then amounts to

$$\int_{\mathbb{S}^{n-1}} \Omega(x) \, \mathrm{d}\sigma(x) = 0$$

This might be starting to look more and more like a homework problem...

For such operators, we leave without proof the fundamental result of Calderón and Zygmund.

Theorem 8. *Let $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$. Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$ and $\int_{\mathbb{S}^{n-1}} \Omega(x) \, \mathrm{d}\sigma(x) = 0$. Define $K(x) = \Omega(x)|x|^{-n}$ and K_ε and T_ε as before. Then there is some constant $A > 0$ depending only on p, q, and n such that*

$$\|T_\varepsilon f\|_{L^p} \leq A \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \|f\|_{L^p}.$$

Furthermore, there is a function Tf such that $T_\varepsilon f \to Tf$ in L^p, and T defined this way is a bounded linear operator on L^p.

This result can be extended even further into Sobolev spaces with a lot more work.