Point-to-set principle for measures

Subhasish Mukherjee

1 Dimension of Measures

Throughout, we try to use r for algorithmic scales and n for dynamical iteration. Other letters are left to the less glamorous indexing.

Let X be a metric space and let μ be a Borel measure on X.

Definition 1.1. Let $x \in X$. The *lower local dimension* and *upper local dimension* at $x \in X$ are defined respectively as

$$\underline{\delta}_x(\mu) = \liminf_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} \quad \text{and} \quad \overline{\delta}_x(\mu) = \limsup_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho}.$$

Definition 1.2. Define the *lower Hausdorff dimension*, *upper Hausdorff dimension*, *lower packing dimension*, and *upper packing dimension* of μ respectively as

$$\begin{aligned}
\underline{\dim}_H(\mu) &= \inf\{\dim_H(E) \mid \mu(E) > 0\} \\
\overline{\dim}^H(\mu) &= \inf\{\dim_H(E) \mid \mu(X \setminus E) = 0\} \\
\underline{\dim}_P(\mu) &= \inf\{\dim_P(E) \mid \mu(E) > 0\} \\
\overline{\dim}^P(\mu) &= \inf\{\dim_P(E) \mid \mu(X \setminus E) = 0\}.
\end{aligned}$$

Fix a countable dense set $Q \subseteq X$ and enumerate $Q = \{q_k\}_{k \in \mathbb{N}}$.

Definition 1.3. Let $A \subseteq \{0,1\}^\omega$ be an oracle, $x \in X$, and $n \in \mathbb{N}$. The *Kolmogorov complexity* of x relative to A at precision r is defined as

$$C_r^A(x) = \min\big\{C^A(k) \;\big|\; x \in B(q_k, 2^{-r})\big\}.$$

The *effective Hausdorff dimension* of x relative to A is defined as

$$\underline{\dim}^A(x) = \liminf_{r \to \infty} \frac{C_r^A(x)}{r}.$$

Note that for each $t \in \mathbb{R}$, the $\big\{x \;\big|\; C_r^A(x) < t\big\} = \bigcup\big\{B(q_k, 2^{-r}) \;\big|\; k \in \mathbb{N}, C^A(k) < t\big\}$ is open so C_r^A is upper semicontinuous and Borel measurable, thus $\underline{\dim}^A$ is also Borel measurable.

Definition 1.4. We say that a collection $\mathcal{B} \subseteq \mathscr{P}(X)$ is a *disjoint collection* if for all $A, B \in \mathscr{B}$ with $A \neq B$, we have that $A \cap B = \varnothing$.

Definition 1.5. We say that $\mathcal{B} \subseteq \mathscr{P}(X)$ is a *Besicovitch cover* of $E \subseteq X$ if each $B \in \mathcal{B}$ is a ball, and for each $x \in E$, there is some $r > 0$ such that $B(x,r) \in \mathcal{B}$. We say that X satisfies the *Besicovitch covering* *property* if there exists some $M \in \mathbb{N}$ such that for all bounded sets $E \subseteq X$ and all Besicovitch covers $\mathcal{B} \subseteq \mathscr{P}(X)$ of E, there are M many disjoint collections $\mathcal{B}_1, \ldots, \mathcal{B}_M \subseteq \mathcal{B}$ such that $E \subseteq \bigcup_{k=1}^{M} \bigcup \mathcal{B}_k$. We say M is the *Besicovitch number* of X.

Hypotheses 1.6. We assume that X is a separable metric space and that X satisfies the Besicovitch covering property with Besicovitch number M. We also assume that μ is a probability measure on X.

First, we recall a standard result on dimensions.

Theorem 1.7. We have

$$\underline{\dim}_H(\mu) = \operatorname*{ess\,inf}_{x\sim\mu} \underline{\delta}_x(\mu)$$

$$\overline{\dim}_H(\mu) = \operatorname*{ess\,sup}_{x\sim\mu} \underline{\delta}_x(\mu)$$

$$\underline{\dim}_P(\mu) = \operatorname*{ess\,inf}_{x\sim\mu} \overline{\delta}_x(\mu)$$

$$\overline{\dim}_P(\mu) = \operatorname*{ess\,sup}_{x\sim\mu} \overline{\delta}_x(\mu)$$

Lemma 1.8. There exists an oracle $A \in \{0,1\}^\omega$ such that for all $x \in X$, we have that $\dim^A(x) \leq \underline{\delta}_x(\mu)$.

Proof. Recall we have a countable dense set $Q = \{q_k\}_{k\in\mathbb{N}} \subseteq X$. For $N \geq 1$, let $E_N = B(q_0, N)$. For each $N \geq 1$ and $r \in \mathbb{N}$, let $\mathcal{B}^{N,r} = \{B(q_k, 2^{-r}) \mid q_k \in Q \cap E_N\}$. Since E_N is open, note that $Q \cap E_N$ is dense in E_N, so for every $x \in E_N$, there is some $q_k \in Q \cap E_N$ such that $d(x, q_k) < 2^{-r}$, which implies that $\mathcal{B}^{N,r}$ covers E_N. By the Besicovitch covering property, there are disjoint subcollections $\mathcal{B}_1^{N,r}, \ldots, \mathcal{B}_M^{N,r} \subseteq \mathcal{B}^{N,r}$ whose union covers E_N. Let $Q_1^{N,r}, \ldots, Q_M^{N,r} \subseteq Q$ respectively be the centers of the balls in $\mathcal{B}_1^{N,r}, \ldots, \mathcal{B}_M^{N,r} \subseteq \mathcal{B}^{N,r}$. Note that disjointness guarantees that for each each $1 \leq j \leq M$, we have that

$$\sum_{q\in Q_j^{N,r}} \mu(B(q, 2^{-r})) \leq 1.$$

Let $\ell_{k,r} = \max\left(1, \lceil -\log\mu(B(q_k, 2^{-r}))\rceil\right)$, so we have that $2^{-\ell_{k,r}} \leq \mu(B(q_k, 2^{-r}))$, which implies that $\sum_{k, q_k\in Q_j^{N,r}} 2^{-\ell_{k,r}} \leq 1$.
By the KC theorem, there are prefix-free binary codes $c_{k,N,r,j} \in 2^{<\omega}$ such that $|c_{k,N,r,j}| = \ell_{k,r}$. Let A be the oracle such that given inputs (N, r, j) and a string w, A returns k if $w = c_{k,N,r,j}$.
To compute $q_k \in Q_j^{N,r}$, the universal machine U^A needs N (encoded in $O(\log N)$ bits), r (encoded in $O(\log r)$ bits), j (encoded in $O(1)$ bits), and the code $c_{k,N,r,j}$ (encoded in $\ell_{k,r} \leq -\log\mu(B(q_k, 2^{-r})) + 1$ bits, so

$$C^A(k) \leq -\log\mu(B(q_k, 2^{-r})) + O(\log N + \log r).$$

For any $x \in E_N$ and $r \in \mathbb{N}$, there is some $q_k \in Q_j^{N,r}$ such that $d(x, q_k) < 2^{-r}$, so $B(x, 2^{-r+1}) \subseteq B(q_k, 2^{-r})$. We thus have for all such x

$$C_r^A(x) \leq C^A(k) \leq -\log\mu\left(B\left(q_k, 2^{-r}\right)\right) + O(\log N + \log r) \leq -\log\mu\left(B\left(x, 2^{-(r+1)}\right)\right) + O(\log N + \log r)$$

Now dividing by r and taking $\liminf_{r\to\infty}$ on both sides, we get the desired result. This holds for all $N \geq 1$, and so it holds for all $x \in X$ as desired. ■

Theorem 1.9. Let (X,d) be a separable metric space that satisfies the Besicovitch covering property. For any finite Borel measure μ on X, we have

$$\underline{\dim}_H(\mu) = \min_{A\in\{0,1\}^\omega} \operatorname*{ess\,inf}_{x\sim\mu} \dim^A(x).$$

Proof. By Lemma 1.8, we have that there exists some oracle A_0 such that $\dim^{A_0}(x) \leq \underline{\delta}_x(\mu)$. Taking essential infimum over $x\sim\mu$ on both sides, we have that $\operatorname{ess\,inf}_{x\sim\mu}\dim^{A_0}(x) \leq \operatorname{ess\,inf}_{x\sim\mu}\underline{\delta}_x(\mu) = \underline{\dim}_H(\mu)$. On the other hand, let $A\in\{0,1\}^\omega$ be any oracle. Let $\alpha = \operatorname{ess\,inf}_{x\sim\mu}\dim^A(x)$. Let $m\in\mathbb{N}$, and set

$$H_m = \{x\in X \mid \dim^A(x) \leq \alpha + 2^{-m}\},$$

so we have $\mu(H_m) \geq 0$. Fix $\gamma > \alpha + 2^{-m}$. Note for $x\in H_m$ that there must be infinitely many $r\in\mathbb{N}$ such that $C_r^A(x) < r\gamma$, so we can find infinitely many $q_k\in Q$ such that for some $r\in\mathbb{N}$ we have $x\in B(q_k,2^{-r})$ and $C^A(k) < r\gamma$. With this in mind, for any $N\in\mathbb{N}$, define the family

$$\mathcal{B}_{R,\gamma} = \{B(q_k,2^{-r}) \mid r\geq R \text{ and } C^A(k) < r\gamma\}.$$

The previous remarks imply that $\mathcal{B}_{R,\gamma}$ covers H_m. Let $\gamma' > \gamma$. We have

$$\sum_{B\in\mathcal{B}_{R,\gamma}} (\operatorname{diam} B)^{\gamma'} \leq \sum_{r=R}^{\infty} \sum_{k, C^A(k)<r\gamma} 2^{-(r+1)\gamma'}.$$

Note that the number of programs shorter than $r\gamma$ is at most $2^{r\gamma}$, so the inner sum is taken over at must $2^{r\gamma}$ many elements. In particular, we have

$$\sum_{B\in\mathcal{B}_{R,\gamma}} (\operatorname{diam} B)^{\gamma'} \leq 2^{\gamma'} \sum_{r=R}^{\infty} 2^{r(\gamma-\gamma')}.$$

As $R\to\infty$, we have that the right hand side goes to zero, which tells us that the γ' Hausdorff measure of H_m is 0 for all $\gamma' > \gamma$. In particular, we have that $\dim_H(H_m) \leq \gamma$, and so $\dim_H(H_m) \leq \alpha + 2^{-m}$. Since $\mu(H_m) > 0$, we have

$$\underline{\dim}_H(\mu) \leq \dim_H(H_m) \leq \alpha + 2^{-m}.$$

This holds for all m, so we have that $\underline{\dim}_H(\mu) \leq \alpha = \operatorname{ess\,inf}_{x\sim\mu}\dim^A(x)$ as desired. ■

2 Entropy

Now suppose that T is a homeomorphism of X that preserves the measure μ. Suppose also that μ is ergodic with respect to T.

Definition 2.1. For $n\in\mathbb{N}$, $x\in X$, and $\varepsilon > 0$, let $B_n(x,\rho) = \{y\in X \mid d(T^jx, T^jy) < \rho \text{ for all } 0\leq j\leq n\}$ be the Bowen ball. Define

$$C_r^A(x,n) = \min\{C^A(k) \mid x\in B_n(q_k,2^{-r})\}.$$

Define the pointwise effective entropies

$$\underline{h}^A(x) = \lim_{r\to\infty} \liminf_{n\to\infty} \frac{1}{n} C_r^A(x,n)$$

$$\overline{h}^A(x) = \lim_{r\to\infty} \limsup_{n\to\infty} \frac{1}{n} C_r^A(x,n)$$

We recall the Brin-Katok theorem.

Theorem 2.2 (Brin-Katok)**.** Suppose that T is a homeomorphism of a compact metric space X and μ is a T-invariant ergodic measure on X. We then have that

$$h_\mu(T) = \lim_{\rho\to 0} \limsup_{n\to\infty} -\frac{\log \mu(B_n(x,\rho))}{n} = \lim_{\rho\to 0} \liminf_{n\to\infty} \frac{\log \mu(B_n(x,\rho))}{n}.$$

Lemma 2.3. There exists an oracle $A \in \{0,1\}^\omega$ such that

$$\operatorname*{ess\,sup}_{x\sim\mu} \overline{h}^A(x) \le h_\mu(T).$$

Proof. Let $\overline{h}_\mu(x,r) = \limsup_n -\frac{1}{n}\log \mu(B_n(x,2^{-r}))$. Note that $\log \mu(B_{n+1}(x,2^{-r})) \supseteq \log \mu(B_n(Tx,2^{-r}))$, so in particular we have $\overline{h}_\mu(x,r) \le \overline{h}_\mu(Tx,r)$. Identical logic for T^{-1} tells us that $\overline{h}_\mu(\cdot,r)$ is T-invariant and thus is equal to a constant almost everywhere, which we call $\overline{h}_\mu(r)$. This and the Brin-Katok theorem tells us that the set

$$\Lambda = \left\{x \in X \;\middle|\; \lim_{r\to\infty} \overline{h}_\mu(x,r) = h_\mu(T) \text{ and } \overline{h}_\mu(x,r) = \overline{h}_\mu(r) \text{ for all } r \in \mathbb{N}\right\}$$

has full measure. Let $x \in \Lambda$. We will endeavor to show that $\overline{h}^A(x) \le h_\mu(T)$, which will imply the result. Fix $\varepsilon > 0$, and pick $r \in \mathbb{N}$ such that for all $r' \ge r$ we have $|\overline{h}_\mu(r') - h_\mu(T)| < \frac{\varepsilon}{2}$. Pick $s \in \mathbb{Q}$ such that $s > \overline{h}_\mu(r+3) > s - \frac{\varepsilon}{2}$. For $n \ge 1$, define the set

$$Q_{n,r,s} = \left\{q_k \in Q \;\middle|\; \mu\left(B_n\left(q_k, 2^{-(r+2)}\right)\right) > 2^{-ns}\right\}$$

and $\mathcal{B}_{n,r,s} = \left\{B_n\left(q_k, 2^{-(r+2)}\right) \;\middle|\; q_k \in Q_{n,r,s}\right\}$. Let $\mathcal{B}'_{n,r,s}$ be a maximal disjoint subfamily of $\mathcal{B}_{n,r,s}$, and $Q'_{n,r,s} \subseteq Q_{n,r,s}$ be the corresponding centers. Note we must have that $\#\mathcal{B}'_{n,r,s} \le 2^{ns}$ so, we can find prefix-free binary codes $c_{k,n,r,s} \in 2^{<\omega}$ such that $|c_{k,n,r,s}| = \lceil ns \rceil$. Let A be the oracle that given inputs (n,r,s) and a string w, A returns k if $w = c_{k,n,r,s}$. In particular, for such (k,n,r,s), we have that $C^A(q_k) \le ns + O_{r,s}(\log n)$.

Since $x \in \Lambda$ we know that $\limsup_n -\frac{1}{n}\log \mu(B_n(x,2^{-(r+3)})) = \overline{h}_\mu(r+3) < s$, so we can find some $N \in \mathbb{N}$ such that for all $n \ge N$ we have $\mu(B_n(x,2^{-r})) > 2^{-ns}$. For such n, we can find $q \in Q$ such that $d_n(x,q) < 2^{-(r+3)}$, so we have that $B_n(x,2^{-(r+3)}) \subseteq B_n(q,2^{-(r+2)})$, which gives that $\mu(B_n(q,2^{-(r+2)})) > 2^{-ns}$, i.e. $q \in Q_{n,r,s}$.

By maximality of $\mathcal{B}'_{n,r,s}$, there must be some $q_k \in \mathcal{B}'_{n,r,s}$ such that $d(q_k,q) < 2^{-(r+1)}$. In particular, $d_n(x,q_k) \le d_n(x,q) + d(q,q_k) < 2^{-r}$. We thus conclude that $C_r^A(x,n) \le ns + O_{r,s}(\log n)$, which after dividing by n, taking $\limsup_n$, and taking $\lim r$, we get that $\overline{h}^A(x) \le s < \overline{h}_\mu(r+3) + \frac{\varepsilon}{2} \le h_\mu(T) + \varepsilon$. ∎

Lemma 2.4. For any oracle $A \in \{0,1\}^\omega$, we have that

$$h_\mu(T) \leq \operatorname*{ess\,inf}_{x\sim\mu} \underline{h}^A(x,T).$$

Proof. Let $\alpha = \operatorname{ess\,inf}_{x\sim\mu} \underline{h}^A(x,T)$. Let $\varepsilon > 0$ and let $\gamma = h_\mu(T) - \varepsilon$.
Let $\underline{h}_\mu(x,r) = \liminf_n -\frac{1}{n}\log\mu(B_n(x,2^{-r}))$. As in Lemma 2.3, we note that $\underline{h}_\mu(x,r)$ is equal to a constant almost everywhere which we call $\underline{h}_\mu(r)$, and

$$\Lambda = \left\{x \in X \;\middle|\; \lim_{r\to\infty} \underline{h}_\mu(x,r) = h_\mu(T) \text{ and } \underline{h}_\mu(x,r) = \underline{h}_\mu(r) \text{ for all } r \in \mathbb{N}\right\}$$

has full measure.
Since $\gamma < h_\mu(T)$, there exists some $r \in \mathbb{N}$ such that for all $r' \geq r$ we have that $\underline{h}_\mu(r') > \gamma$. Let $\delta = \frac{1}{2}\left(\underline{h}_\mu(r) - \gamma\right) > 0$. For $N \in \mathbb{N}$, set $G_N = \left\{x \in X \;\middle|\; \mu(B_n(x,2^{-r})) \leq 2^{-n\left(\underline{h}_\mu(r)-\delta\right)} \text{ for all } n \geq N\right\}$. Note we have that $\bigcup_{N\in\mathbb{N}} G_N \supseteq \Lambda$, so $\mu(G_N) \to 1$ as $N \to \infty$.
For $n \in \mathbb{N}$, set $E_n = \left\{x \in X \;\middle|\; C^A_{r+1}(x,n) \leq n\gamma\right\}$. Let $S_n = \left\{q_k \in Q \;\middle|\; C^A(k) \leq n\gamma\right\}$, and note that $E_n \subseteq \bigcup_{q\in S_n} B(q,2^{-(r+1)})$. Additionally, there are at most $2^{n\gamma}$ possible outputs of programs of length at most $n\gamma$, so $\#S_n \leq 2^{n\gamma}$.
We will aim to estimate $\mu(E_n \cap G_N)$. Let $q \in S_n$, and if $B_n(q,2^{-(r+1)}) \cap G_N \neq \varnothing$ and $N \geq N$, pick $y \in B_n(q,2^{-(r+1)}) \cap G_N$. We then get that

$$\mu(B_n(q,2^{-(r+1)})) \leq \mu(B_n(y,2^{-r})) \leq 2^{-n(\underline{h}_\mu(r)-\delta)} = 2^{-n(\gamma+\delta)}.$$

In particular, we get

$$\mu(E_n \cap G_N) \leq \mu\left(\bigcup_{q\in S_n} B_n(q,2^{-(r+1)}) \cap G_N\right) \leq \#S_n 2^{-n(\gamma+\delta)} \leq 2^{-n\delta}.$$

In particular, we have $\sum_{n\geq N}\mu(E_n\cap G_N) < \infty$. By Borel-Cantelli, $\mu\left(\bigcap_{N\geq1}\bigcup_{n\geq N}(E_n\cap G_N)\right) = 0$. Since $\mu(G_N) \to 1$, we must have that $\mu\left(\bigcap_{N\geq1}\bigcup_{n\geq N}E_n\right) = 0$. In particular, we conclude that for μ almost every $x \in X$, we have that $C^A_{r+1}(x,n) > n\gamma$ for n large enough, which implies that $\underline{h}^A(x) \geq \gamma = h_\mu(T) - \varepsilon$. Since ε was arbitrary, the result follows. ∎

Combining the two lemmas, we get

Theorem 2.5. With the above hypotheses, we have

$$h_\mu(T) = \min_A \operatorname*{ess\,inf}_{x\sim\mu} \underline{h}^A(x) = \min_A \operatorname*{ess\,sup}_{x\sim\mu} \overline{h}^A(x).$$