Complemented Subspaces

Subhasish Mukherjee

February 1, 2022

Contents

Quotient Spaces

Definitions and Preliminaries

I start with some definitions for completeness.

Definition 1 (Coset)**.** Let V be a vector space and W be a subspace of V. Then for $v \in V$, we define the coset

$$v + W = \{v + w \mid w \in W\}$$

We will also write $v + W$ as $[v]_W$.

Definition 2 (Quotient Space)**.** Let V be a vector space over a field $\mathbb{F}$ and $W \subseteq V$ be a subspace of V. Then we define the *quotient space* V/W as the set of cosets

$$V/W = \{v + W \mid v \in V\} = \{[v]_W \mid v \in V\}.$$

For $[v]_W, [u]_W \in V/W$ and $c \in \mathbb{F}$, we also define

$$[v]_W + [u]_W = [v+u]_W \text{ and } c[v]_W = [cv]_W$$

We now state some propositions, the verifications of which are not too difficult and thus omitted because I am lazy.

Proposition 3. *Let $[v]_W, [u]_W \in V/W$. Then $[v]_W = [u]_W \Leftrightarrow v - u \in W$.*

Remark 4. Proposition 3 show that the quotient space V/W is the same object one would obtain by quotienting by the equivalence relation $\sim$ where $v \sim w$ if and only if $v - u \in W$.

Proposition 5. *Addition and scalar multiplication on V/W as described in Definition 2 is well-defined - i.e. if $[v]_W, [v']_W, [u]_W, [u']_W \in V/W$ and $c \in \mathbb{F}$ with $[v]_W = [v']_W$ and $[u]_W = [u']_W$, then $[v]_W + [u]_W = [v']_W + [u']_W$ and $c[v]_W = c[v']_W$.*

Proposition 6. *The quotient space V/W equipped with addition and scalar multiplication as described in Definition 2 is a vector space.*

Norms on Quotient Spaces

For a normed vector space V, we move to transfer the norm structure to the quotient space.

Definition 7. Let V be a normed vector space with norm $\|\cdot\|_V$ and W a subspace of V. Define $\|\cdot\|_{V/W}$ via

$$\|[v]_W\|_{V/W} = \inf\{\|v+w\|_V \mid w \in W\}$$

We don't yet know whether this definition is indeed a norm. The next proposition shows that this is the case when our subspace W is closed.

Proposition 8. *If W is a closed subspace of V, then the map $\|\cdot\|_{V/W}$ defined in Definition 7 is a norm on V/W. Furthermore, for all $v \in V$ we have that $\|[v]_W\|_{V/W} \leq \|v\|_V$.*

Proof. Clearly $\|[v]_W\|_{V/W} \geq 0$ for all $[v]_W \in V/W$, establishing nonnegativity. We also see $\|[0]_W\|_{V/W} = 0$. Conversely, suppose $\|[v]_W\|_{V/W} = 0$. Then by definition of the norm, there exists a sequence $\{w_n\} \subseteq W$ such that $\lim_{n\to\infty} \|v + w_n\|_V = 0$ which implies $\lim_{n\to\infty} w_n = -v$. Because W is closed, this implies $-v \in W$ and so $v \in W$, i.e. $[v]_W = [0]_W$ as desired.

Remark 9. In establishing postive definiteness, the necessity of W being closed is shown. If W was not closed, then we would get a semi-norm on V/W. Quotienting to obtain a norm would give an isometrically isomorphic object to $V/\overline{W}$ with the norm defined in Definition 7.

Now let $c \in \mathbb{F}$. Then to establish homogeneity, we see

$$\begin{aligned}
\|c[v]_W\|_{V/W} &= \inf\{\|cv+w\|_V \mid w \in W\} \\
&= \inf\{\|c(v+w)\|_V \mid w \in W\} \\
&= \inf\{|c|\,\|v+w\|_V \mid w \in W\} \\
&= |c| \inf\{\|v+w\|_V \mid w \in W\} \\
&= |c|\,\|[v]_W\|_{V/W}
\end{aligned}$$

Finally, let $[v]_W, [u]_W \in V/W$. Let $\varepsilon > 0$. Pick $w_1, w_2 \in W$ such that $\|v+w_1\|_V < \|[v]_W\|_{V/W} + \frac{\varepsilon}{2}$ and $\|u+w_2\|_V < \|[u]_W\|_{V/W} + \frac{\varepsilon}{2}$. Then we see

$$\begin{aligned}
\|[v+u]_W\|_{V/W} &\leq \|v+u+w_1+w_2\|_V \\
&\leq \|v+w_1\|_V + \|u+w_2\|_V \\
&< \|[v]_W\|_{V/W} + \|[u]_W\|_{V/W} + \varepsilon
\end{aligned}$$

This holds for all $\varepsilon > 0$, so we have $\|[v+u]_W\|_{V/W} \leq \|[v]_W\|_{V/W} + \|[u]_W\|_{V/W}$ as desired, establishing the triangle inequality and the fact that $\|\cdot\|_{V/W}$ is indeed a norm. For the last point, we can just note that $0 \in W$ and so $\|[v]_W\|_{V/W} \leq \|v+0\|_V = \|v\|_V$ by the definition of $\|\cdot\|_{V/W}$. ∎

In the following results, we assume that V is a vector space and W is a closed subspace of V. We equip V/W with the norm in Definition 7. With the norm in hand, we can check completeness of our quotient space. Before that, we prove a few quick lemmas.

Lemma 10. *Let $[v]_W, [u]_W \in V/W$ and $\varepsilon > 0$ be arbitrary. Then there exists $v' \in [v]_W$ such that $\|v' - u\|_V < \|[v]_W - [u]_W\|_{V/W} + \varepsilon$.* [1]

[1] Importantly, we are not touching u, setting up the proof for our theorem.

Proof. By definition of the infimum, we know there exists $w \in W$ such that

$$\|v - u + w\|_V < \|[v-u]_W\|_{V/W} + \varepsilon = \|[v]_W - [u]_W\|_{V/W} + \varepsilon.$$

We let $v' = v + w$ and we see that $v' - v = w \in W$ and so $v' \in [v]_W$, thus we have our desired result. ∎

Lemma 11. *The projection map $[\cdot]_W : V \to V/W$ is continuous.*

Proof. Let $\varepsilon > 0$ and $v \in V$. Let $u \in B_V(v, \varepsilon)$. Then we see that

$$\|[v]_W - [u]_W\|_{V/W} = \|[v-u]_W\|_{V/W} \leq \|v - u\|_V < \varepsilon$$

and so $[\cdot]_W$ is continuous as desired. ∎

Theorem 12. *If V is a Banach space, then so is V/W.*

Proof. We already know V/W is a vector space, so it suffices to show completeness. Towards that end, let $\{[v_n]_W\}_{n=\ell}^{\infty} \subseteq V/W$ be Cauchy. We may extract a subsequence and assume without loss of generality that $\|[v_{n+1}]_W - [v_n]_W\|_{V/W} < \frac{1}{2^{n+1}}$. We pick $v'_\ell \in [v_\ell]_W$. Then for $n \geq \ell$, by Lemma 10 we can recursively pick $v'_{n+1} \in [v_{n+1}]_W$ such that $\left\|v'_{n+1} - v'_n\right\|_V < \frac{1}{2^n}$. Then $\{v'_n\}_{n=\ell}^{\infty}$ is a Cauchy sequence in V, and so converges to some $v \in V$. Then by Lemma 11 we have $\lim_{n\to\infty} [v_n]_W = [v]_W$, finishing the proof. ∎

Complementing Subspaces

Definition and Preliminary

In the last section we built up some analysis in quotient spaces. In this section, we examine a related notion - complements.

Definition 13 (Subspace Complement)**.** Suppose V is a Banach space and W is a closed subspace of V. Then we say that W *splits* or is *complemented* if there exists another closed subspace $X \subseteq V$ such that $V = W \oplus X$. [2] In this case, we say X is the *complement* of W.

[2] Recall that we say $V = W \oplus X$ when for all $v \in V$ there exists unique $x \in X$ and $w \in W$ such that $v = x + w$.

Proposition 14. *Suppose V is a Banach space and W is a closed subspace of V with complement X. Then V/W is isomorphic to X.*

Remark 15. In fact, the map in Proposition 14 is also isometric, but I was too lazy to even finish the proof so it would be optimistic to presume that I'd show this too.

Proof. Let $v \in V$. We see that $V = W \oplus X$ also implies there exists some $x \in X$ such that $x \in [v]_W$. To establish uniqueness, suppose there exists $x, x' \in X$ such that $x, x' \in [v]_W$. Then we would have $x = v - w$ and $x' = v - w'$ for some $w, w' \in W$ and so $v = x + w = x' + w'$, which in turn implies $x = x'$ again because $V = W \oplus X$. Thus for each $v \in V$ there is unique $x \in X$ such that $x \in [v]_W$. Then define the map $\varphi : V/W \to X$ by sending $[v]_W$ to this unique element $x \in X$ such that $x \in [v]_W$. One can then verify that [3] φ is indeed well-defined and an isomorphism. ∎

[3] Translated to mean "I didn't want to write it up, but"

Nonexistence of the Complement of c_0 in ℓ^∞

Proposition 14 establishes a relation between complements and quotient spaces. However, W need not split for V/W to be a Banach space. In finite dimensional spaces, every subspace is closed and splits, but in infinite dimensional spaces, this may not hold. Indeed, we already have examples of subspaces that are not closed - the subspace $\mathcal{P}([0,1];R) \subseteq C_b^0([0,1];\mathbb{R})$ of polynomials is not closed as $\overline{\mathcal{P}([0,1];R)} = C_b^0([0,1];\mathbb{R}) \neq \mathcal{P}([0,1];R)$. As the main result of this section will show, we can also have closed subspaces that don't split.

Lemma 16. *There is an uncountable set I and a family $\{A_i\}_{i\in I}$ of subsets of $\mathbb{N}$ such that each A_i is infinite and $A_j \cap A_i$ is finite for all $i, j \in I$ with $i \neq j$.*

Proof. By bijecting, we can see that it suffices to prove the result for $\mathbb{Q}$. Then take I to be $\mathbb{R} \setminus \mathbb{Q}$. Then for $i \in I$, we let $A_i \subseteq \mathbb{Q}$ be a sequence of rationals converging to i. By uniqueness of limits, we must have that $A_i \cap A_j$ is finite for all $i \neq j$ as desired. ■

Lemma 17. *Suppose $T : \ell^\infty \to \ell^\infty$ is a linear operator with $\ker(T) \supseteq c_0$. Then there is an infinite subset A of $\mathbb{N}$ such that $T(x) = 0$ for all $x \in \ell^\infty$ with $\operatorname{supp}(x) \subseteq A$.* [4]

[4] We are thinking of ℓ^∞ as the space of bounded functions from $\mathbb{N}$ to $\mathbb{R}$, so $\operatorname{supp}(x) = \{n \in \mathbb{N} \mid x(n) \neq 0\}$.

Proof. Take a family $\{A_i\}_{i\in I}$ as in Lemma 16. Towards a contradiction, suppose that our result didn't hold, so for each $i \in I$, there exists $x_i \in \ell^\infty$ supported on A_i with $Tx_i \neq 0$. Because T is linear, we normalize x_i and assume that $\|x_i\|_{\ell^\infty} = 1$.

Define $I_m = \{i \in I \mid (Tx_i)(m) \neq 0\}$. Then because I is uncountable, there must be $n \in \mathbb{N}$ with I_n uncountable.[5] Then we define $I_{n,\ell} = \left\{ i \in I \;\middle|\; |(Tx_i)(n)| \geq \frac{1}{\ell} \right\}$. Again because I_n is uncountable, we know there exists some $k \in \mathbb{N}$ such that $I_{n,k}$ is uncountable.

[5] Here we are using uncountability of I - if I was merely countable, there would be no guarantee I_n is infinite for some n.

Let $J \subseteq I_{n,k}$ be an arbitrarily large finite set. Define $y = \sum_{j\in J} x_j \frac{|(Tx_j)(n)|}{(Tx_j)(n)}$. Then we see

$$(Ty)(n) = \sum_{j\in J} |(Tx_j)(n)| \geq \frac{|J|}{k}$$

by our definition of $I_{n,k}$. Define $D = \bigcup_{i,j\in J} (A_i \cap A_j)$. Because $A_i \cap A_j$ is finite for all $i, j \in J$, we know D is finite. We can also note that $|(Ty)(n)| \leq 1$ for all $i \notin D$. [6] Thus we can write $y = f + z$ with f supported on D and $\|z\|_{\ell^\infty} \leq 1$. Then $T(f) = 0$ by hypothesis on T and so $T(y) = T(z)$ which implies $\|T(y)\| \leq \|T\| \|z\| \leq \|T\|$ and so $|J| \leq \|T\| k$, contradicting that $I_{n,k}$ is infinite. ■

[6] This follows by our normalization condition and that we only get contribution from at most one element of the sum outside of D.

Theorem 18. *The closed subspace $c_0(\mathbb{N};\mathbb{R}) = \{x \in \ell^\infty(\mathbb{N};\mathbb{R}) \mid x(n) \to 0 \text{ as } n \to \infty\}$ does not split in $\ell^\infty(\mathbb{N};\mathbb{R})$.*

Proof. Suppose for the sake of contradiction that there was some closed subspace $W \subseteq \ell^\infty$ with $\ell^\infty = c_0 \oplus W$. Then define the projection map $P : \ell^\infty \to c_0$ by writing $x \in \ell^\infty$ uniquely as $x = c + w$ for $c \in c_0$, $w \in W$ and saying $P(x) = c$. Then we see that $\ker(I - P) = c_0$, thus applying Lemma 17 there is some infinite subset A of $\mathbb{N}$ with $(I - P)(x) = 0$ for all $x \in \ell^\infty$ with $\operatorname{supp}(x) \subseteq A$. Then we see that $\chi_A \notin c_0$ because A is infinite. But $\operatorname{supp}(\chi_A) = A$, so we have $P(\chi_A) = \chi_A$ implying χ_A is in c_0, a contradiction. Thus c_0 cannot split in ℓ^∞. ■

Complementing Subspaces in Hilbert Spaces

Complements might not exist in general Banach spaces, but they do in Hilbert spaces! To prove this, we first require a key result.

Remark 19. An even cooler result holds - a Banach space is a Hilbert space if and only if every subspace splits! The proof was more advanced and I did not understand it but you can look up "On the Complemented Subspace Problem" by J. Lindenstrauss and L. Tzafiri if you're interested.

Lemma 20. *Suppose V is a Hilbert space and C is a closed convex set in V. Then there exists a unique $v \in C$ such that $\|v\|_V = \inf\{\|x\|_V \mid x \in C\}$.*

Proof. Let $d = (\inf\{\|x\|_V \mid x \in C\})^2$. Then we can find a sequence $\{v_n\}_{n=1}^{\infty}$ such that $d \le \|v_n\|_V^2 < d + \frac{1}{n}$ and so $\|v_n\|_V^2 \to d$ as $n \to \infty$. By convexity of C, we know that $\frac{v_n+v_m}{2} \in C$ for all $n, m \ge 0$ and so $\left\|\frac{v_n+v_m}{2}\right\|_V^2 \ge d$. Then by the parallelogram law,

$$\begin{aligned}\left\|\frac{v_n - v_m}{2}\right\|_V^2 &= 2\left\|\frac{v_n}{2}\right\|_V^2 + 2\left\|\frac{v_m}{2}\right\|_V^2 - \left\|\frac{v_n + v_m}{2}\right\|_V^2 \\ &< \frac{d}{2} + \frac{1}{2n} + \frac{d}{2} + \frac{1}{2m} - d = \frac{1}{2}\left(\frac{1}{n} + \frac{1}{m}\right)\end{aligned}$$

and so $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence in V which implies it has some limit v. Continuity of the norm then implies that $d = \lim_{n\to\infty} \|v_n\|_V^2 = \|v\|_V^2$, establishing existence.
For uniqueness, suppose we had $w \in C$ with $\|v\|^2 = \|w\|^2 = d$. Then

$$\left\|\frac{v-w}{2}\right\|_V^2 = 2\left\|\frac{v}{2}\right\|_V^2 + 2\left\|\frac{w}{2}\right\|_V^2 - \left\|\frac{v+w}{2}\right\|_V^2 \le \frac{d}{2} + \frac{d}{2} - d = 0$$

and so $v = w$ as desired. ■

Lemma 21. *Suppose that $W \subsetneq V$ is a closed subspace of V. Then there exists a nonzero element $v_0 \in V$ with $v_0 \perp W$.* [7]

[7] Recall $v_0 \perp W$ means that $\langle v_0, w\rangle = 0$ for all $w \in W$.

Proof. Let $v \in V \setminus W$. Then the set $v - W$ is convex and closed, so by Lemma 20 it contains a unique element $v_0 = v - w_0 \in v - W$ of minimum norm. Because W is closed and $v \notin W$, we know that $v_0 \neq 0$, so it suffices to prove $v_0 \perp W$.
Because v_0 is of minimal norm in $v - W$, for any $w \in W$ and $\lambda \in \mathbb{F}$, we have

$$\|v_0\|_V = \|v - w_0\|_V \le \|v - w_0 + \lambda w\|_V = \|v_0 + \lambda w\|_V$$

and so subtracting and simplifying, we have $2\mathrm{Re}(\lambda\langle w, v_0\rangle) + |\lambda|^2 \|w\|_V^2 \ge 0$. Then setting $\lambda = a\langle v_0, w\rangle$ for some $a \in \mathbb{R}$, we have $a\,|\langle w, v_0\rangle|^2\,(2 + a\|w\|_V^2) \ge 0$ for all $w \in W$ and $a \in \mathbb{R}$. However, this means $|\langle w, v_0\rangle|^2$ must be 0 as we can pick a and w such that $-\frac{2}{\|w\|^2} < a < 0$ which would make the expression negative, contradicting our analysis. Thus $\langle w, v_0\rangle = 0$ for all $w \in W$ and so $v_0 \perp w$ as desired. ■

Theorem 22. *If V is a Hilbert space and W is a closed subspace of V, then $W^\perp$ is closed and $V = W \oplus W^\perp$, and so every closed subspace of V splits.*

Proof. To start, let us show that $W^\perp$ is closed. Towards that end, let $\{u_n\}_{n=\ell}^{\infty} \subseteq W^\perp$ be convergent to some u in V. Then for any $w \in W$, we have that $\lim_{n\to\infty} \langle u_n, w\rangle = 0$. By continuity of the inner product, we see that we can pull the limit in so we get $\langle u, w\rangle = 0$. This holds for all $w \in W$, so $u \in W^\perp$ and thus $W^\perp$ is closed as desired.
We can note that $W \cap W^\perp = \{0\}$, so we are justified in writing $W \oplus W^\perp$. Let us show that $W \oplus W^\perp$ is closed. Suppose $\{w_n + u_n\}_{n=\ell}^{\infty} \subseteq W \oplus W^\perp$ is convergent to some $v \in V$

Remark 23. The proof that $W^\perp$ is closed did not rely on W being closed to begin with - in fact for general subspaces $W \subseteq V$, it's not too hard to show that $\left(W^\perp\right)^\perp = \overline{W}$

with $\{w_n\}_{n=\ell}^{\infty} \subseteq W$ and $\{u_n\}_{n=\ell}^{\infty} \subseteq W^{\perp}$. Then by the Pythagorean theorem,

$$\|(w_n + u_n) - (w_m + u_m)\|_V^2 = \|w_n - w_m\|_V^2 + \|u_n - u_m\|_V^2$$

and thus $\{w_n\}_{n=\ell}^{\infty}$ and $\{u_n\}_{n=\ell}^{\infty}$ are Cauchy and thus convergent to w and u in W and $W^{\perp}$ respectively. Then we see that $v = w + u$ and so $v \in W \oplus W^{\perp}$ as desired.

Now suppose for the sake of contradiction that $W \oplus W^{\perp} \neq V$. Then by Lemma 21, there exists a nonzero element $v_0 \in V$ with $v_0 \perp (W \oplus W^{\perp})$. Then $v_0 \perp W$ and $v_0 \perp W^{\perp}$ which implies $v_0 \in W^{\perp} \cap W$ and so $v_0 = 0$, a contradiction. Thus $W \oplus W^{\perp} = V$ as desired, finishing the proof. ■

References

The main reference for these notes was *Banach Spaces and Differential Calculus* by Abraham, Marsden, and Ratiu, 1998. The proof that c_0 does not admit a complement in ℓ^{∞} was adapted from this Math Stack Exchange post.